Using the Green's Function Method to Calculate Pressure Fluctuations in Compressible Multifluids

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Abstract

In this paper, we present a pressure equation describing fluctuations in a turbulent mixing layer between two fluids. In the comoving frame of the mixing layer, the pressure fluctuation satisfies a decaying wave equation that can be solved analytically using the Green's function method. The obtained 1-D analytic solution for pressure fluctuations across the mixing layer displays the desired features required by the BHR turbulence transport model. It is shown that the pressure fluctuations, generated by shocks or instabilities in the mixing region, decay exponentially away from the mixing layer. This new solution could provide a theoretical foundation for the current artificial nonlocal length-scale equation used in the BHR model. The solutions successfully reduce to the well known incompressible form in the limit of large sound speed.

1. Introduction

Small perturbations in a multifluid system produce buoyancy and shear driven instabilities at an interface between distinct fluids. These instabilities grow and develop into a turbulent mixing layer in which the velocity fluctuations induce departures from pressure equilibrium and excite sound waves in the surrounding fluids. These effects play an important role in the dynamical evolution of compressible fluids and have to be taken into account even if the fluids are near pressure equilibrium. Quantitatively analyzing these effects is fundamental to the studies of turbulent multifluid mixing and is crucial to the construction of successful mixing models exhibiting a monotone density profile across the mixing layer. Green's function method has been very effective in getting the solutions for pressure fluctuations in incompressible fluids[1,2] or potential flows. In this paper, we propose to use Green's function method to evaluate the possible pressure fluctuations in compressible fluid systems.

2. Dynamic Equations

We begin with the Navier-Stokes mass and momentum conservation equations:

$$\frac{\partial m}{\partial t} + \vec{\nabla} \cdot (m\vec{v}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(m\vec{v}) + \vec{\nabla} \cdot (m\vec{v}^2 + \mathbf{\Pi} + P\boldsymbol{\delta}) = 0, \tag{2}$$

where m is mass density, $m\vec{v}$ is momentum density, P is pressure, $\mathbf{\Pi}$ is viscous stress, and $\boldsymbol{\delta}$ is the unit dyadic.

Averaging equations (1) and (2) gives the bulk fluid equations, which can be written as[3]:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{u}) = 0, \tag{3}$$

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \vec{\nabla} \cdot (\rho\vec{u}^2 + \boldsymbol{R} + \bar{\mathbf{\Pi}} + \bar{P}\boldsymbol{\delta}) = 0, \tag{4}$$

where $\rho = \langle m \rangle$, $\rho\vec{u} = \langle m\vec{v} \rangle$, $\boldsymbol{R} \equiv \langle m(\vec{v} - \vec{u})^2 \rangle$, $\bar{\mathbf{\Pi}} \equiv \langle \mathbf{\Pi} \rangle$, and $\bar{P} = \langle P \rangle$. The angle brackets denote some suitable average.

To describe the fluctuations within a turbulent mixing layer, it is convenient to transform to the comoving frame of the mixing layer. Let $\vec{x}(t) \equiv \vec{x}_0 + \int_0^t dt_0 \vec{u}(\vec{x}_0, t_0)$ be the time-dependent position of a fluid element at the center of mass of the mixing layer. We do the coordinate transformation:

$$\vec{r}' = \vec{r} - \vec{x}(t), \quad t' = t,$$

under which

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \dot{\vec{x}} \cdot \vec{\nabla}', \quad \vec{\nabla} = \vec{\nabla}', \quad \dot{\vec{x}} \equiv \frac{\partial \vec{x}}{\partial t}\Big|_{t=t'} = \vec{u}\big(\vec{x}(t), t\big)\Big|_{t=t'}. \tag{5}$$

Then neglecting the viscous stress, Eqs. (1)–(4) become

$$\frac{\partial m}{\partial t'} - \dot{\vec{x}} \cdot \vec{\nabla}' m + \vec{\nabla}' \cdot (m\vec{v}) = 0, \tag{6}$$

$$\frac{\partial}{\partial t'}(m\vec{v}) - \dot{\vec{x}} \cdot \vec{\nabla}'(m\vec{v}) + \vec{\nabla}' \cdot (m\vec{v}\vec{v}) + \vec{\nabla}' P = 0, \tag{7}$$

$$\frac{\partial \rho}{\partial t'} - \dot{\vec{x}} \cdot \vec{\nabla}' \rho + \vec{\nabla}' \cdot (\rho\vec{u}) = 0, \tag{8}$$

$$\frac{\partial}{\partial t'}(\rho\vec{u}) - \dot{\vec{x}} \cdot \vec{\nabla}'(\rho\vec{u}) + \vec{\nabla}' \cdot (\rho\vec{u}\vec{u} + \mathbf{R}) + \vec{\nabla}' \bar{P} = 0. \tag{9}$$

Subtracting Eq. (8) from Eq. (6) and Eq. (9) from Eq. (7), gives

$$\frac{\partial m'}{\partial t'} + \vec{\nabla}' \cdot (m\vec{v}' + m'\vec{u}') = 0, \tag{10}$$

$$\begin{aligned} \frac{\partial (m\vec{v}' + m'\vec{u})}{\partial t'} &- \dot{\vec{x}} \cdot \vec{\nabla}'(m\vec{v}' + m'\vec{u}) \\ &+ \vec{\nabla}' \cdot (m\vec{u}\vec{v}' + m\vec{v}'\vec{u} + m'\vec{u}\vec{u} + \mathbf{R}') + \vec{\nabla}' P' = 0, \end{aligned} \tag{11}$$

where $m' \equiv m - \rho$, $\vec{v}' \equiv \vec{v} - \vec{u}$, $\vec{u}' \equiv \vec{u} - \dot{\vec{x}}$, $P' \equiv P - \bar{P}$, and $\boldsymbol{R}' \equiv m\vec{v}'^2 - \boldsymbol{R}$ are fluctuating quantities. Note that

$$m\vec{v} - \rho\vec{u} = m\vec{v}' + m'\vec{u},$$

$$m\vec{v}\vec{v} - \rho\vec{u}\vec{u} - \mathbf{R} = m\vec{u}\vec{v}' + m\vec{v}'\vec{u} + m'\vec{u}\vec{u} + \mathbf{R}'.$$

The main effects of compressibility on a flow are the propagation of sound waves and the formation of shocks, where the latter can be treated as spatial discontinuities described by jump conditions on the fluid properties. Outside these discontinuities, the gradients of the bulk flow quantities are presumed small. Taking the divergence of each term in Eq. (11) and applying Eq. (10) yields

$$-\frac{\partial^2 m'}{\partial t'^2} + \ddot{\vec{x}} \cdot \vec{\nabla}' m' + \vec{\nabla}'^2 : \big(m\vec{u}'\vec{v}' + m\vec{v}'\vec{u}' + m'\vec{u}'\vec{u}' + \mathbf{R}' \big) + \nabla'^2 P' = 0. \quad (12)$$

We now consider simple fluids (*e.g.* isothermal or adiabatic) for which density and pressure fluctuations are approximately related by

$$dP' \simeq C_s^2 dm', \quad (13)$$

where $C_s^2 \equiv \frac{\partial P}{\partial m}$ is the local sound speed. For more complex fluids, Eq. (13) can be generalized to a more complicated equation of state by supplementing Eqs. (1) and (2) with the energy conservation equation. Substituting Eq. (13) into Eq. (12) and letting

$$\mathbf{\Sigma}' \equiv m\vec{u}'\vec{v}' + m\vec{v}'\vec{u}' + m'\vec{u}'\vec{u}' + \mathbf{R}' \quad (14)$$

represent the source term, we finally obtain a wave equation for P'

$$\frac{1}{C_s^2} \frac{\partial^2 P'}{\partial t'^2} - \frac{\ddot{\vec{x}}}{C_s^2} \cdot \vec{\nabla}' P' - \nabla'^2 P' = \vec{\nabla}'^2 : \mathbf{\Sigma}', \quad (15)$$

where the temporal variation of the sound speed in the comoving frame has been neglected. This wave equation (with propagation speed C_s) describes the evolution of P' in compressible fluids as the propagation of a decaying sound wave driven by a source. The force term on the right side of the equation is responsible for any generation of fluctuations in the pressure by the turbulent flow. Outside the turbulent region, this force is negligible, but pressure fluctuations nevertheless occur due to sound propagation. This type of equation is also encountered when describing the propagation of electromagnetic waves in conducting media. Shocks can be included by treating their jump conditions as internal boundary restraints tying together different solutions to Eq. (15). If $\dot{\vec{x}}(t) \simeq \vec{u}$, $\vec{u}' = 0$, the source term is simplified to $\mathbf{\Sigma}' = \mathbf{R}'$. Furthermore, if the spatial variation of the bulk fluid is negligible,

$$\ddot{\vec{x}} \equiv -\big[\frac{1}{\rho}\vec{\nabla}' \cdot (\mathbf{R} + \bar{P}\delta)\big]_{\vec{r}'=0} \sim 0.$$

Then Eq. (15) reduces to a simple wave equation

$$\frac{1}{C_s^2}\frac{\partial^2 P'}{\partial t'^2} - \nabla'^2 P' = \vec{\nabla}'^2 : \mathbf{\Sigma}'. \tag{16}$$

In the limit of incompressible fluids, $C_s \to \infty$, this equation reduces to Poisson's equation:

$$\nabla'^2 P' \simeq -\vec{\nabla}' \cdot \vec{\sigma}', \tag{17}$$

where $\vec{\sigma}' \equiv \vec{\nabla} \cdot \mathbf{\Sigma}'$.

3. The Green's functions

As in electrodynamics, the inhomogeneous wave equation (15) can be solved in terms of a Green's function G which satisfies homogeneous boundary conditions and a causality condition:

$$G(\vec{r}, t|\vec{r}_0, t_0) = 0 \qquad \text{if} \quad t < t_0, \tag{18}$$

where we have dropped the primes from $\vec{r}$ and t. The corresponding Green's function for Eq. (15) satisfies

$$-\frac{1}{C_s^2}\frac{\partial^2 G}{\partial t^2} - \frac{\ddot{\vec{x}}}{C_s^2} \cdot \vec{\nabla}' G - \nabla'^2 G = 4\pi\delta(\vec{r} - \vec{r}_0)\delta(t - t_0). \tag{19}$$

It is reasonable to assume that the initial G and $\partial G/\partial t$ should be zero for $t < t_0$; that is, if an impulse occurs at t_0, no effects of the impulse should be present at an earlier time. Thus the initial conditions that G satisfies are

$$\big[G\big]_{t=0} = 0, \quad \big[\frac{\partial G}{\partial t}\big]_{t=0} = 0. \tag{20}$$

For convenience, we let $\vec{k} \equiv \ddot{\vec{x}}/C_s^2$ which is assumed varying very slowly with space and time, $\tau \equiv t - t_0$, $\vec{s} \equiv \vec{r} - \vec{r}_0$ and

$$G = G^*(s, \tau) e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}.$$

Substituting G into Eq. (19) gives a scalar wave equation in 3-D,

$$\frac{1}{C_s^2}\frac{\partial^2 G^*}{\partial t^2} - \nabla^2 G^* + \frac{1}{4}k^2 G^* = 4\pi\delta(\vec{s})\delta(\tau). \tag{21}$$

We introduce $G^*_\omega \equiv g e^{-i\omega\tau}$, where g satisfies the following equation

$$\big[\nabla^2 + \frac{\omega^2}{C_s^2} - \frac{1}{4}k^2\big] g = -4\pi\delta(\vec{s}). \tag{22}$$

Noticing that $\delta(\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} d\omega$, we obtain the solution to Eq. (22)

$$g = \frac{1}{s} e^{-i\sqrt{(\omega/C_s)^2 - (\frac{k}{2})^2} s}.$$

which, in turn, leads to

$$G^*(s,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*_\omega d\omega = \frac{1}{2\pi s}\int_{-\infty}^{\infty} e^{i(\sqrt{(\omega/C_s)^2 - (\frac{k}{2})^2} s - \omega\tau)} d\omega. \tag{23}$$

Rewriting

$$G^*(s,\tau) = \frac{1}{s}\frac{\partial h(s,\tau)}{\partial s}, \tag{24}$$

where

$$h(s,\tau) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i(\sqrt{(\omega/C_s)^2 - (k/2)^2} s - \omega\tau)}}{\sqrt{(\omega/C_s)^2 - (k/2)^2}} d\omega \quad . \tag{25}$$

Using the Cauchy's Integral Theorem and Formula for h gives

$$h(s,\tau) = -C_s J_0\big[\frac{kC_s}{2}\sqrt{\tau^2 - (s/C_s)^2}\big]\varpi(\tau - s/C_s), \tag{26}$$

where

$$\varpi(\tau - s/C_s) \equiv \begin{cases} 1 & \text{for } s < C_s\tau, \\ 0 & \text{for } s > C_s\tau, \end{cases}$$

and

$$J_0\big[\frac{kC_s}{2}\sqrt{\tau^2 - (s/C_s)^2}\big] \equiv -\frac{1}{2\pi i}\int_0^{2\pi} e^{-i\frac{kC_s}{2}\sqrt{\tau^2 - (s/C_s)^2}\cos\theta} d\theta$$

is the Bessel function of the first kind for integer order $n = 0$.

Thus, the final solution to G^* in 3-D is

$$G^*_{3D}(s,\tau) = \frac{\delta(\tau - \frac{s}{C_s})}{s} - \frac{k}{2\sqrt{\tau^2 - (\frac{s}{C_s})^2}} J_1\big[\frac{kC_s}{2}\sqrt{\tau^2 - (\frac{s}{C_s})^2}\big]\varpi(\tau - \frac{s}{C_s}) \quad , \tag{27}$$

where $J_0(0) = 1$, $\partial J_0(x)/\partial x = -J_1(x)$, and $\partial \varpi(x)/\partial x = \delta(x)$ have been used, and J_1 is the Bessel function of the first kind for integer order $n = 1$.

In two and one dimensional case, the G^*s can be calculated as

$$\begin{aligned} G^*_{2D}(s,\tau) &= \int_{-\infty}^{\infty} G^*_{3D}(s,\tau) d\eta \\ &= \frac{2}{\sqrt{C_s^2\tau^2-\rho^2}}\varpi(\tau-\frac{s}{C_s}) - 2\int_{-\infty}^{\infty}\frac{J_1(z)\varpi(\tau-\frac{s}{C_s})}{\sqrt{C_s^2\tau^2-\rho^2-z^2}}dz \\ &= \frac{2}{\sqrt{C_s^2\tau^2-\rho^2}}\varpi(\tau-\frac{s}{C_s})\big\{1+2\sinh^2\big[\frac{k}{4}\sqrt{C_s^2\tau^2-\rho^2}\big]\big\}, \end{aligned} \tag{28}$$

and

$$G^*_{1D}(s,\tau) = -h(s,\tau) = C_s J_0\big[\frac{kC_s}{2}\sqrt{\tau^2-(s/C_s)^2}\big]\varpi(\tau - s/C_s), \tag{29}$$

where $s^2 = \eta^2 + \rho^2$.

Substituting the obtained G^*_i, $i = \text{3D}, \text{2D}, \text{1D}$, into G, we finally obtain the Green's function as

$$G_i(s,\tau) \equiv e^{-\frac{1}{2}\vec{k}\cdot\vec{s}} G^*_i(s,\tau), \quad i = \text{3D},\ \text{2D},\ \text{1D},$$

that is, for 3-D,

$$G_{3D}(s,\tau) = e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}\Big\{\frac{\delta(\tau-\frac{s}{C_s})}{s} - \frac{k/2}{\sqrt{\tau^2-(\frac{s}{C_s})^2}}J_1\big[\frac{kC_s}{2}\sqrt{\tau^2-(\frac{s}{C_s})^2}\big]\varpi\Big\}, \tag{30}$$

for 2-D,

$$G_{2D}(s,\tau) = \frac{2e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}}{\sqrt{C_s^2\tau^2-\rho^2}}\varpi(\tau-\frac{s}{C_s})\big\{1+2\sinh^2\big[\frac{k}{4}\sqrt{C_s^2\tau^2-\rho^2}\big]\big\}, \tag{31}$$

and for 1-D,

$$G_{1D}(s,\tau) = e^{-\frac{1}{2}\vec{k}\cdot\vec{s}} C_s J_0\big[\frac{kC_s}{2}\sqrt{\tau^2-(s/C_s)^2}\big]\varpi(\tau - s/C_s), \tag{32}$$

Clearly, the Green's function in 3-D compressible fluid contains two terms: the first term in (30) is a production of the initial pulse, reduced, however, by two factors. The first, $1/s$, is the geometrical factor which appeared in the solution of the simple wave equation. The second is the factor $e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}$ which tells us that this part of the wave, generated by the point source, decays with time as it moves through the medium. The second term in Eq.(30) constitutes the wake. If $|\vec{k}| \sim 0$ or $|\vec{k}|L \ll 1$ (L is the dimension of the source region or the width of mixing layer), the second term can be neglected and Eq. (30) further reduces to

$$G(|\vec{r}-\vec{r}_0|,\tau) = e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}\frac{\delta(\tau - \frac{|\vec{r}-\vec{r}_0|}{C_s})}{|\vec{r}-\vec{r}_0|}. \tag{33}$$

This represents a decaying retarded potential which describes that the effect of an impulse at a time t_0 is felt at a distance s away at a time $t = t_0 + s/C_s$, and this effect decays exponentially with the distance.

The two and one dimensional Green's functions are accordingly have expressions as follows

$$G_{2D}(s,\tau) = \frac{2e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}}{\sqrt{C_s^2\tau^2-\rho^2}}\varpi(\tau - \frac{s}{C_s}) \tag{34}$$

and

$$G_{1D}(s,\tau) = C_s e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}. \tag{35}$$

A striking difference between the two and the three dimensional cases is that in three dimensions the effect of an impulse after a time τ has elapsed will be found concentrated on a sphere of radius $s = C_s\tau$ whose center is at the source point. This is a virtue of the function $\delta[(s/C_s)-\tau]$ which occurs in three dimensions. In two dimensions, the effect at a time τ due to an impulsive source is spread over the entire region $s < C_s\tau$ because the singularity at $s = C_s\tau$ in two dimensional cases is very weak when compared with the δ function singularity in the three dimensional case. In one dimensional situations, the effect of an impulse delivered at a time t_0 at the point r_0 is not concentrated at the point $|r-r_0| = \pm C_s(t-t_0)$ but rather exists throughout the region of extent $2C_s(t-t_0)$ with the source point r_0 at the middle point.

In the incompressible limit, $C_s \to \infty$, $k \to 0$, the three-dimensional Green's function (30), as expected, is reduced to

$$G(|\vec{r}-\vec{r}_0|,\tau) = \frac{\delta(\tau)}{|\vec{r}-\vec{r}_0|}, \tag{36}$$

a well-known potential for the Poisson's equation, while the two dimensional Green's function remains a constant and the one dimensional Green's function has a discontinuity in slope equal to -4π at the point source x_0, where identity $\delta(C_s\tau - s) = \frac{1}{C_s}\delta(\tau - s/C_s)$ is used.

Clearly, from here we see that the effect of compressibility is to change the Green's function from a Poisson potential to a retarded potential. It is shown that the dynamics of incompressible fluid is very similar to electrostatics, *i.e.*, the interaction between two points in both systems is instantaneous and the speed of propagation is infinite, while the dynamics of compressible fluid is like electrodynamics where the interaction between two points is retarded and propagated with a finite speed. However, the dynamics of compressible fluid is more complicated than electrodynamics because of the very different properties of sound and light speed.

4. Pressure fluctuations

In terms of the Green's function, using the Green's theorem and Eq. (20), we calculate

$$\int\int \big[G(\vec{r},t)\times(13)(\vec{r}_0,t_0) - P'(\vec{r},t)\times(16)(\vec{r}_0,t_0)\big]dV_0 dt_0$$

and obtain the solution for P' in (15) by Green's theorem

$$\begin{aligned} P'(\vec{r},t) = \frac{1}{4\pi}\int_0^t dt_0 dV_0 \Big[G\big(\vec{\nabla}_0^2 : \mathbf{\Sigma}'\big) + \frac{\vec{k}}{C_s^2}\cdot\big(G\vec{\nabla}_0 P' - P'\vec{\nabla}_0 G\big)\Big] \\ \frac{1}{4\pi}\int_0^t dt_0 \oint d\vec{S}_0\cdot\big(G\vec{\nabla}_0 P' - P'\vec{\nabla}_0 G\big) \end{aligned} \tag{37}$$

where dV_0 represents the element of volume of the considered domain and $d\vec{S}_0$ is the element of the vector surface area surrounding the domain. The first two terms of Eq. (37) represent the effects of volume sources, while the last term expresses the boundary conditions, where we have used the initial condition on P'_0 and $[\partial P'/\partial t]_0$ of P' and $\partial P'/\partial t$. In principle, for any given source (v') and boundary condition, the pressure fluctuation P' is calculated uniquely. Apparently, due to the decay factor $e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}$ in the Green's function, for any kind of source, the pressure fluctuation will decay with the distance from the source.

For clearness, now we take 1-D planar geometry as an example. In 1-D, $s \equiv y - y_0$, $\tau \equiv t - t_0$, then the 1-D Green's function is expressed as

$$G_{1D}(s,\tau) = C_s e^{-\frac{1}{2}k(y-y_0)} J_0\big[\frac{kC_s}{2}\sqrt{\tau^2 - (\frac{y-y_0}{C_s})^2}\big]\varpi(\tau - \frac{y-y_0}{C_s}), \quad (38)$$

and

$$J_0\big[\frac{kC_s}{2}\sqrt{\tau^2 - (\frac{y-y_0}{C_s})^2}\big] \equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i\frac{kC_s}{2}\sqrt{\tau^2-(\frac{y-y_0}{C_s})^2}\cos\theta}, \quad (39)$$

Let $\xi \equiv \frac{kC_s}{2}\sqrt{\tau^2 - (\frac{y-y_0}{C_s})^2}$, for $|\xi| < \pi$, J_0 can be expressed by series expansion

$$J_0 = \sum_0^\infty (-1)^k \frac{1}{(k!)^2}\big(\frac{\xi}{2}\big)^{2k}. \quad (40)$$

In the case of $kL/2 \ll 1$, Eq. (40) reduces to

$$J_0 \approx 1 - \frac{\xi}{4} = 1 - \frac{kC_s}{16}\left[\frac{kC_s}{2}\sqrt{\tau^2 - (\frac{y-y_0}{C_s})^2}\right]. \quad (41)$$

Therefore the 1-D Green's function for planar geometry becomes

$$G_{1D} \approx C_s e^{-\frac{1}{2}k(y-y_0)}\left\{1 - \frac{kC_s}{16}\left[\frac{kC_s}{2}\sqrt{\tau^2 - (\frac{y-y_0}{C_s})^2}\right]\right\}. \quad (42)$$

In a comoving frame, the second term in G_{1D} is very small compared to the first one, therefore as a zeroth order approximation, the 1D Green's function can be written as

$$G_{1D} \sim C_s e^{-\frac{1}{2}k(y-y_0)}. \quad (43)$$

This is just the kernel function or "nonlocal function" $e^{-b(y-y_0)}$ used by other authors [5], but here the parameter "b" can be uniquely determined.

Substituting this expression into Eq. (37), we have

$$\begin{aligned} P'(\vec{y},t) &= \frac{1}{4\pi}\bigg(\int_0^t dt_0 \int_{V_0} dV_0 G \frac{\partial^2 \Sigma'}{\partial y_0^2} + \int dt_0 \oint d\vec{S}_0 \cdot \big[G\frac{\partial P'}{\partial y_0} - P'\frac{\partial G}{\partial y_0}\big]\bigg) \\ &\equiv I_1 + I_2, \end{aligned} \quad (44)$$

where

$$I_1 \equiv \frac{1}{4\pi}\int_0^t dt_0 \int_{V_0} dV_0 G\frac{\partial^2\Sigma'}{\partial y_0^2} \tag{45}$$

describes the contribution from the sources Σ' inside the domain, and

$$I_2 \equiv \frac{1}{4\pi}\int dt_0 \oint d\vec{S}_0\cdot\big[G\frac{\partial P'}{\partial y_0} - P'\frac{\partial G}{\partial y_0}\big]\Big) \tag{46}$$

denotes the contributions from the boundaries.

In order to evaluate the pressure fluctuation P', we first evaluate the integral I_1, *i.e.*,

$$I_1 = \frac{C_s}{4\pi}\int dy_0 e^{-\frac{1}{2}k(y-y_0)}\int d\theta dt_0 e^{-i\frac{kC_s}{2}\sqrt{\tau^2-(\frac{y-y_0}{C_s})^2}\cos\theta}\frac{\partial^2\Sigma'}{\partial y_0^2}. \tag{47}$$

To evaluate this integral, we would like to consider two special cases: (i) the near field, $C_s\tau \gg (y-y_0)$; and (ii) the far field, $C_s\tau \ll (y-y_0)$.

(i) The near field, $C_s\tau \gg (y-y_0)$. In this physical situation, the observation point is close to the source, thus

$$e^{-i\frac{kC_s}{2}\sqrt{\tau^2-(\frac{y-y_0}{C_s})^2}\cos\theta} \sim e^{-i\frac{kC_s\cos\theta}{2}\tau}.$$

If the source term is not explicitly a function of t_0, substituting (33) into (32), we obtain

$$\int_0^{2\pi} d\theta \int_0^t d\tau e^{-i\frac{kC_s\cos\theta}{2}\tau}\frac{\partial^2\Sigma'}{\partial y_0^2} = A(t)\frac{\partial^2\Sigma'}{\partial y_0^2}, \tag{48}$$

where

$$A(t) \equiv \int_0^{2\pi} d\theta \int_0^t d\tau e^{-i\frac{kC_s\cos\theta}{2}\tau}$$

is a function of only time. Thus, substituting the above into I_1, after many times integration, it gives

$$\begin{aligned} I_1 = -\frac{A(t)}{4\pi}\Bigg\{&\left[e^{-\frac{1}{2}k(y-y_0)}\frac{\partial\Sigma'}{\partial y_0}\right]_{y_0^-}^{y_0^+} - \frac{k}{2}\left[e^{-\frac{1}{2}k(y-y_0)}\Sigma'\right]_{y_0^-}^{y_0^+} \\ &+\Big(\frac{k}{2}\Big)^2\int_{y_0^-}^{y_0^+} dy_0\Sigma' e^{-\frac{1}{2}k(y-y_0)}\Bigg\}, \end{aligned} \tag{49}$$

and this finally leads to

$$P'(\vec{y},t) = -\frac{A(t)}{4\pi}\Big\{\Big[e^{-\frac{1}{2}k(y-y_0)}\frac{\partial\Sigma'}{\partial y_0}\Big]_{y_0^-}^{y_0^+} - \frac{k}{2}\Big[e^{-\frac{1}{2}k(y-y_0)}\Sigma'\Big]_{y_0^-}^{y_0^+} \tag{50}$$

$$+\big(\frac{k}{2}\big)^2\int_{y_0^-}^{y_0^+} dy_0\Sigma' e^{-\frac{1}{2}k(y-y_0)}\Big\} - \frac{1}{4\pi}\int dt_0\Big[G\frac{\partial P'}{\partial y_0} - P'\frac{\partial G}{\partial y_0}\Big]_{y_0^-}^{y_0^+},$$

where y_0^- and y_0^+ are, respectively, the positions of the lower and upper edge of the mixing zone, and

$$k\Sigma' \equiv 2mu'v'/C_s^2 + m'(u'/C_s)^2 + R'/C_s^2.$$

For subsonic fluctuations, u', $v' \ll C_s$, $k\Sigma'$ is very small, the first term in Eq. (50) dominates. Therefore, in 1-D planar case, for any given $\mathbf{\Sigma}'$, the pressure fluctuations generated by the fluid mixing indeed decay exponentially with the distance from the mixing layer. This agrees with the DNS calculations and the experimental data. Similar calculations can be done for spherical case.

(ii) The far field. In this case, $C_s^2\tau^2 - (y-y_0)^2 \ll C_s^2\tau^2$, and y is far from the source. Thus

$$e^{-i\frac{kC_s}{2}\sqrt{\tau^2-(\frac{y-y_0}{C_s})^2}\cos\theta} \sim 1 - i\frac{kC_s}{2}\sqrt{\tau^2 - (\frac{y-y_0}{C_s})^2}\cos\theta,$$

and

$$\int_0^{2\pi} d\theta e^{-i\frac{kC_s}{2}\sqrt{\tau^2-(\frac{y-y_0}{C_s})^2}\cos\theta} \sim \int_0^{2\pi} d\theta(1 - i\frac{kC_s}{2}\sqrt{\tau^2-(\frac{y-y_0}{C_s})^2}\cos\theta) = 2\pi.$$

This leads to

$$I_1 = -\frac{1}{2C_s}\int dt_0\int dy_0 e^{-\frac{k}{2}(y-y_0)}\frac{\partial^2\Sigma'}{\partial y_0^2}.$$

From the calculations in case (i), we obtain

$$\int dy_0 e^{-\frac{k}{2}(y-y_0)}\frac{\partial^2\Sigma'}{\partial y_0^2} = \Big[e^{-\frac{1}{2}k(y-y_0)}\frac{\partial\Sigma'}{\partial y_0}\Big]_{y_0^-}^{y_0^+} - \frac{k}{2}\Big[e^{-\frac{1}{2}k(y-y_0)}\Sigma'\Big]_{y_0^-}^{y_0^+} + \big(\frac{k}{2}\big)^2\int_{y_0^-}^{y_0^+} dy_0\Sigma' e^{-\frac{1}{2}k(y-y_0)}. \tag{51}$$

Therefore, integral I_1 can be calculated as

$$I_1 = \frac{1}{2C_s}\int dt_0 \left[e^{-\frac{1}{2}k(y-y_0)}\frac{\partial \Sigma'}{\partial y_0}\right]_{y_0^-}^{y_0^+} - \frac{k}{2}\left[e^{-\frac{1}{2}k(y-y_0)}\Sigma'\right]_{y_0^-}^{y_0^+} + \left(\frac{k}{2}\right)^2 \int_{y_0^-}^{y_0^+} dy_0 \Sigma' e^{-\frac{1}{2}k(y-y_0)}. \tag{52}$$

Substituting I_1 into P', we have

$$P'(\vec{y},t) = \frac{1}{2C_s}\int dt_0 \left[e^{-\frac{1}{2}k(y-y_0)}\frac{\partial \Sigma'}{\partial y_0}\right]_{y_0^-}^{y_0^+} - \frac{k}{2}\left[e^{-\frac{1}{2}k(y-y_0)}\Sigma'\right]_{y_0^-}^{y_0^+} + \left(\frac{k}{2}\right)^2 \int_{y_0^-}^{y_0^+} dy_0 \Sigma' e^{-\frac{1}{2}k(y-y_0)} - \frac{1}{4\pi}\int dt_0 \left[G\frac{\partial P'}{\partial y_0} - P'\frac{\partial G}{\partial y_0}\right]_{y_0^-}^{y_0^+}. \tag{53}$$

Similarly, the last two terms in Eq. (53) are very small comparing to the first two terms for subsonic fluctuations.

It is shown that in both the near field solution (50) and the far field solution (53), if the last two terms are negligible for $a\tau \ll C_s$, P' actually is expressed as a sort of jumping condition cross the mixing layer with an exponential decay factor.

5. Conclusion

In this paper, we derived a diffused wave equation for pressure fluctuations in compressible fluid in a comoving frame from the Navier-Stokes equations. This equation reduces to the simple wave equation if the fluid is less compressible, and to a Poisson equation if the fluid is incompressible. Also we have derived the Green's functions for the diffused wave equation in 1-D, 2-D, and 3-D respectively. All Green's functions show an exponential decay behavior $e^{-\frac{1}{2}\vec{k}\cdot\vec{s}}$ with the distance from the mixing layer. This provides a theoretical foundation for the current use of an artificial nonlocal Kernel function in BHR model. In incompressible limit, the 3-D Green's function reduces to the solution of Poisson equation (a potential of a point source) as expected. Last, we have provided an analytic expression for the pressure fluctuations in compressible fluid by using the Green's function method. Applications to the 1-D planar case is presented. In 1-D planar case, the pressure fluctuations generated by mixing indeed decays exponentially with the distance from the mixing layer. This agrees with both DNS calculations and experimental data. Similar calculations can be done for spherical case. In particular, we have examined the solutions, respectively, in the cases of near field and far field. We found that in compressible multifluids the pressure fluctuations at a time t and a point $\vec{r}$ are actually caused by the fluctuations of the local mass densities and velocities at a prior time $t-(s/c)$ and at a point which is a distance s away from $\vec{r}$. These fluctuations decay exponentially with the distance from the source (*i.e.,* the interface between the fluids or mixing layer). When the speed of sound approaches infinity, all of our solutions derived for compressible fluids successfully reduce to the well known forms for incompressible fluids, where, the pressure fluctuations are induced instantaneously by the fluctuations of the mass densities and velocities everywhere.

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