Rate of growth of the linear Richtmyer-Meshkov instability

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RM instability**:** *perturbation growth due to the vorticity generation at both sides of a contact surface by means of corrugated shock fronts.*

We distinguish two situations:

1) A shock is reflected back or,

2) a rarefaction is reflected instead

Two possible approaches to get the growth rate are:

1-heuristic models based on an impulsive formulation (RM, VMG, MB),

2-rigorous linear theory:

-numerical solution as in Richtmyer, CPaAM 13, 297 (1960) or in Yang, Zhang and Sharp (PoF 6, 1856 (1994)),
- series expansions as in Velikovich, PoF 8, 1666 (1996), Wouchuk and Nishihara, PoP 3, 3761 (1996).
-closed analytical formulas deduced from linear theory [Fraley, PoF 29, 376 (1986) Wouchuk and Nishihara, PoP 3,3761 (1996);4, 1028 (1997)]

It is clear that an estimate by means of a closed formula would be useful be it for numerical or experimental studies.

It is always possible to get asymptotic expressions of the rate of growth truncating the temporal expansions. However, the validity of such expressions is limited to very weak incident shocks.

The same restriction applies to any heuristic approach like the ones based on an impulsive formulation of the instability.

Before discussing an exact formulation that leads to a very accurate determination of the growth rate, we discuss some of the hypothesis underlying the heuristic models.

The impulsive formulations make the assumptions:

a) ***Incompressible*** *fluid flow after shock transit, and*

b) ***Irrotational*** *perturbations left by the shocks.*

The general form of the vorticity profile left by a rippled shock is:

$$\delta\omega(x,y) \quad = \quad \Omega \quad \delta p(x, t = \frac{x}{U_s})\sin(ky)$$

$$\delta\omega(x,y) \quad = \quad \Omega \quad \delta p_s(kx/\sinh(\theta_s))\sin(ky)$$

$$\delta\omega(x,y) \quad = \quad \sum_{n\geq 0} A_{2n+1} J_{2n+1}\left(kx/\sinh(\theta_s)\right) \quad \sin(ky)$$

We see that vorticity production could be substantial, and this fact depends on fluids properties as well as on the incident shock intensity.

The vorticity left by the shock fronts is the memory of the effect of the "rippled compression" of the fluid elements.

For a strictly planar shock the flow behind it will be trivially irrotational.

For a corrugated front, the flow behind it will be rotational.

The vorticity deposited in the interior of the fluids will have a definite effect on the interface growth as we are going to see. It can not be neglected for strong shocks or highly compressible fluids.

Tangential velocity:

$$\frac{\partial v_y}{\partial t} = \delta p_i(t) \xrightarrow[0+]{\int^{\infty} \ldots dt} \rho_{af}(\delta v_{ya}^{\infty} - \delta v_{ya}^{0}) = \rho_{bf}(\delta v_{yb}^{\infty} - \delta v_{yb}^{0})$$

Normal velocity:

$$\frac{\partial \delta v_x}{\partial t} = -\frac{1}{\rho_{mf}} \frac{\partial \delta p_i}{\partial x} \xrightarrow[0+]{\int^{\infty} \ldots dt} \delta v_i^{\infty}$$

$$\delta v_i^{\infty}, \quad \delta v_{ya}^{\infty}, \quad \textit{and} \quad \delta v_{yb}^{\infty}$$

are not equal in the general case.

In fact, the parameters:

$$F_a = \delta v_{ya}^{\infty} + \delta v_i^{\infty},$$
$$F_b = \delta v_{yb}^{\infty} - \delta v_i^{\infty}$$

are different from zero and become important for very strong shocks and highly compressible fluids.

How do we relate the parameters F_a and F_b with the shock compression history?

Asymptotic velocity fields:

-Incompressibility:

$$\frac{d\delta v_x}{dx} + \delta v_y = 0,$$

-Vorticity:

$$\frac{d\delta v_y}{dx} + \delta v_x = \Omega \delta p(x/\sinh\theta_s)$$

We get, for example:

$$\frac{d^2 \delta v_x}{dx^2} - \delta v_x = -\Omega \delta p(x / \sinh\theta_s)$$

The right hand side can not be made zero for moderate to strong shocks.

In fact, not only the qualitative form of the velocity profile is modified because of the bulk vorticity: the asymptotic value of the interface velocity could be seriously inferior to any value deduced from an irrotational assumption.

*An **exact expression** of the linear asymptotic growth rate is obtained:*

$$\delta v_i^{\infty} = \frac{\rho_{bf}\,\delta v_{yb}^{0} - \rho_{af}\,\delta v_{ya}^{0}}{\rho_{bf} + \rho_{af}} + \frac{-\rho_{bf} F_b + \rho_{af} F_a}{\rho_{bf} + \rho_{af}}$$

F_a and F_b must be related with the compressible evolution of the instability.

For very weak shocks we have the scaling laws:

$$\begin{cases} F_a & \propto & (M_t^2 - 1)^{7/2} \\ F_b & \propto & (M_r^2 - 1)^{7/2} \end{cases} \qquad \begin{cases} \delta v_{ya}^0 & \propto & (M_t^2 - 1) \\ \delta v_{ya}^0 & \propto & M_r^2 - 1 \end{cases}$$

And then, an irrotational estimation for the asymptotic velocity is justified:

$$\delta v_{irrot}^{\infty} = \frac{\rho_{bf}\delta v_{yb}^0 - \rho_{af}\delta v_{ya}^0}{\rho_{bf} + \rho_{af}}$$

We make a Laplace transform of the equation for δv_x and get:

$$\delta V_{xb}(\sigma) \quad = \quad \frac{\sigma\,\delta v_i^{\infty} - \delta v_{yb}^{\infty} - \Omega_b \sinh\theta_r \delta P_r(\sigma \sinh\theta_r)}{\sigma^2 - 1}$$

A similar equation holds in fluid "a".

The function δP is the time Laplace transform of the shock front pressure perturbations.

To get bounded perturbations, we see that it must be:

$$F_b = -\delta v_i^{\infty} + \delta v_{yb}^{\infty} = \Omega_b \sinh\theta_r \delta P_r(\sinh\theta_r)$$

And an analogous relationship holds in fluid "a".

If a rarefaction were reflected, then: $F_b = 0$.

Thus, to get the values of F_a *and* F_b*, we need temporal averages of the shock pressure functions.*

How do we calculate the parameters $\boldsymbol{F_a}$ ***and*** $\boldsymbol{F_b}$***?***

We change to the coordinate system:

$$\begin{cases} kx = r\sinh\theta \\ kc_f t = r\cosh\theta \end{cases}$$

The shock-fronts coordinates are defined by:

$$\tanh\theta_t = -\frac{U_t}{c_{af}} \qquad \tanh\theta_r = \frac{U_r}{c_{bf}}$$

It can also be seen that the Laplace transform of the shock pressure in each fluid can be written as:

$$\delta P_m(\theta, q) = \frac{F_{m1}(q-\theta) + F_{m2}(q+\theta)}{\cosh q}$$

Where "q" is related to the Laplace variable "s" through:

$$s = \sinh q$$

In principle we have four unknown functions:

$$\begin{cases} F_{a1}(q_a), F_{a2}(q_a) \\ F_{b1}(q_b), F_{b2}(q_b) \end{cases}$$

After some algebra, at the shock fronts and at the interface, we can relate Fa1 and Fb2 in the following way:

$$\begin{cases} F_{a1}(q_a) + \phi_{b3} F_{b2}(q_b) = \phi_{b1} + \phi_{b2} F_{b2}(q_b + 2\theta_r) \\ F_{b2}(q_b) + \phi_{a3} F_{a1}(q_a) = \phi_{a1} + \phi_{a2} F_{a1}(q_a - 2\theta_t) \end{cases}$$

Besides, it can be seen that the desired parameters F_a and F_b are easily related to specific values of F_{a1} and F_{b2}:

$$\begin{cases} F_a = \varepsilon_{a1} F_{a1}(-2\theta_t) + \varepsilon_{a0} \\ F_b = \varepsilon_{b1} F_{b2}(2\theta_r) + \varepsilon_{b0} \end{cases}$$

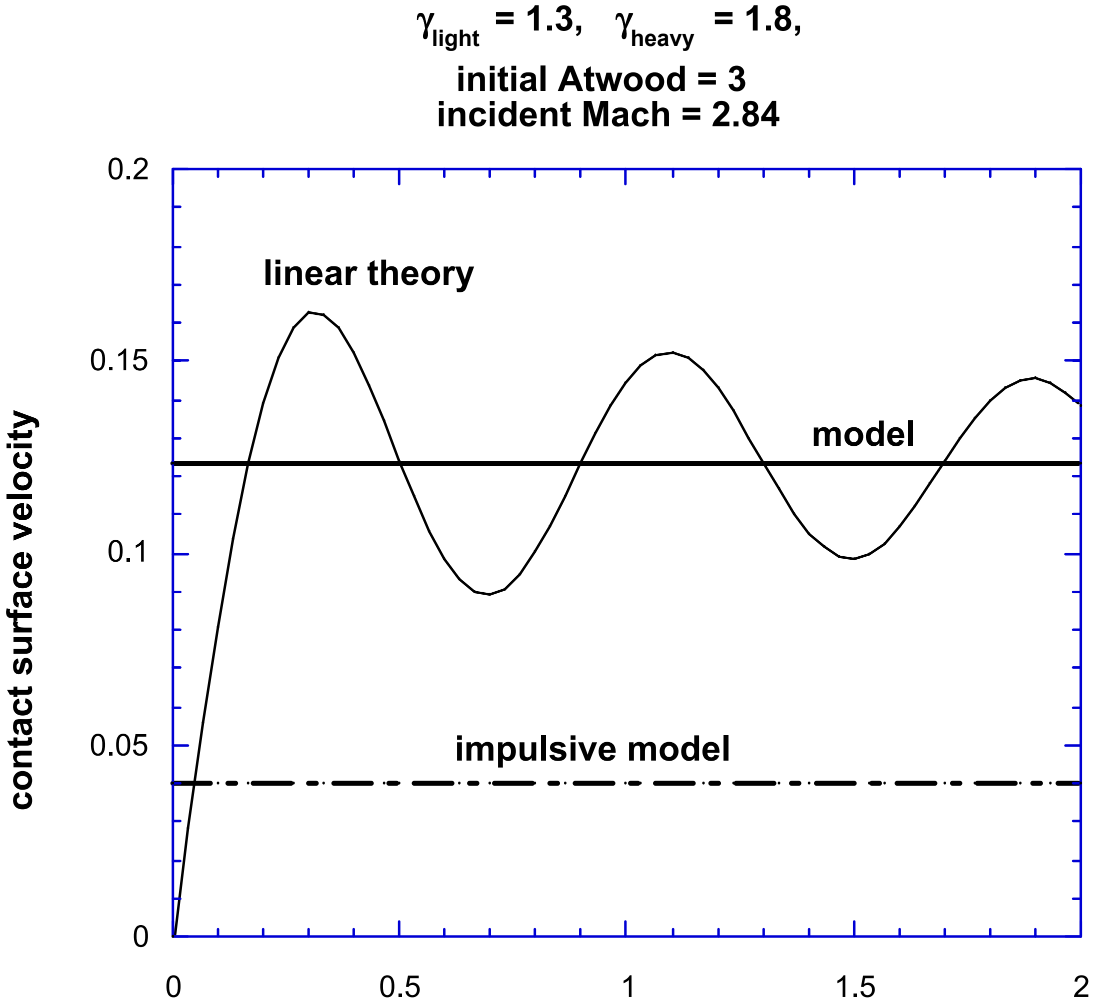
A <u>very fast and accurate</u> algorithm can be implemented to get F_a and F_b.

We define an iteration sequence:

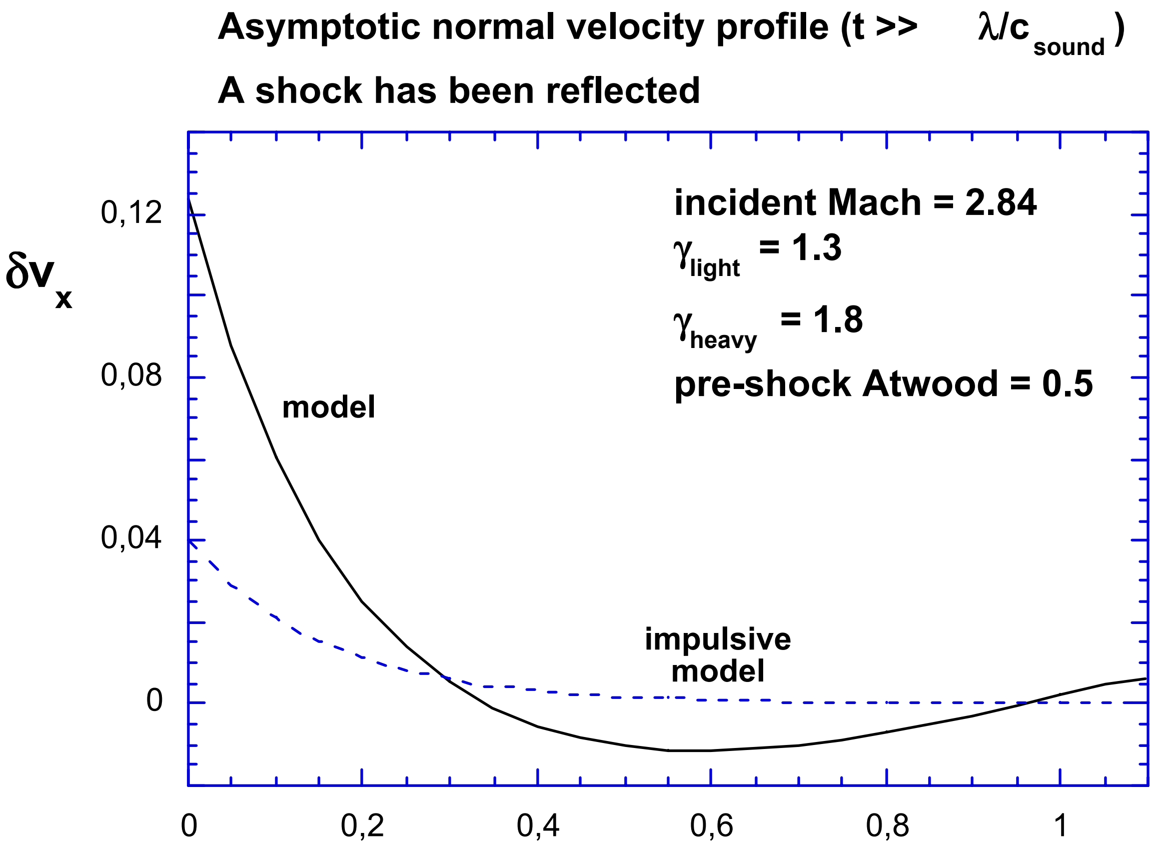
$$F_a^{[n]} \qquad \textit{and} \qquad F_b^{[n]}$$

with which we get the parameters F_a and F_b.

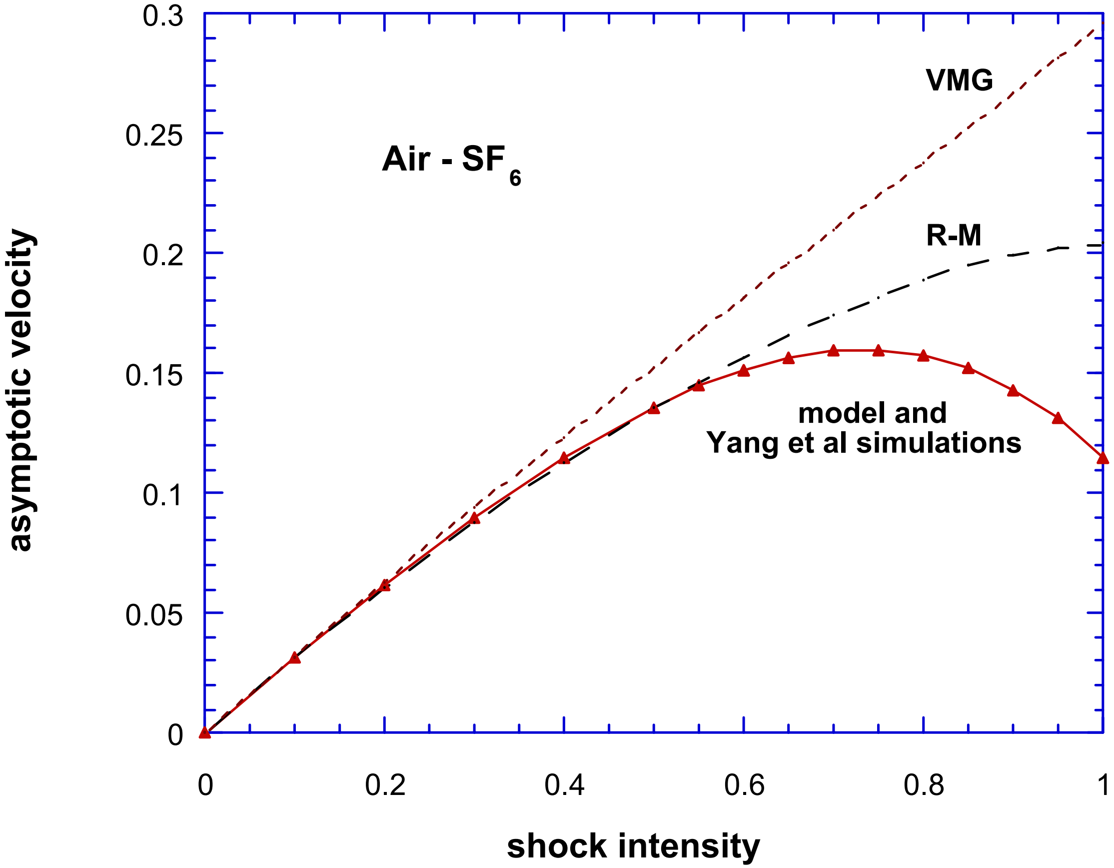
With just the starting values (n = 0) we can get up to 3 digits in the asymptotic velocities even for very strong shocks and highly compressible fluids.
For details see Phys. Plasmas ***8****, 2890 (2001), Phys. Rev. E 63, 056303 (2001).*



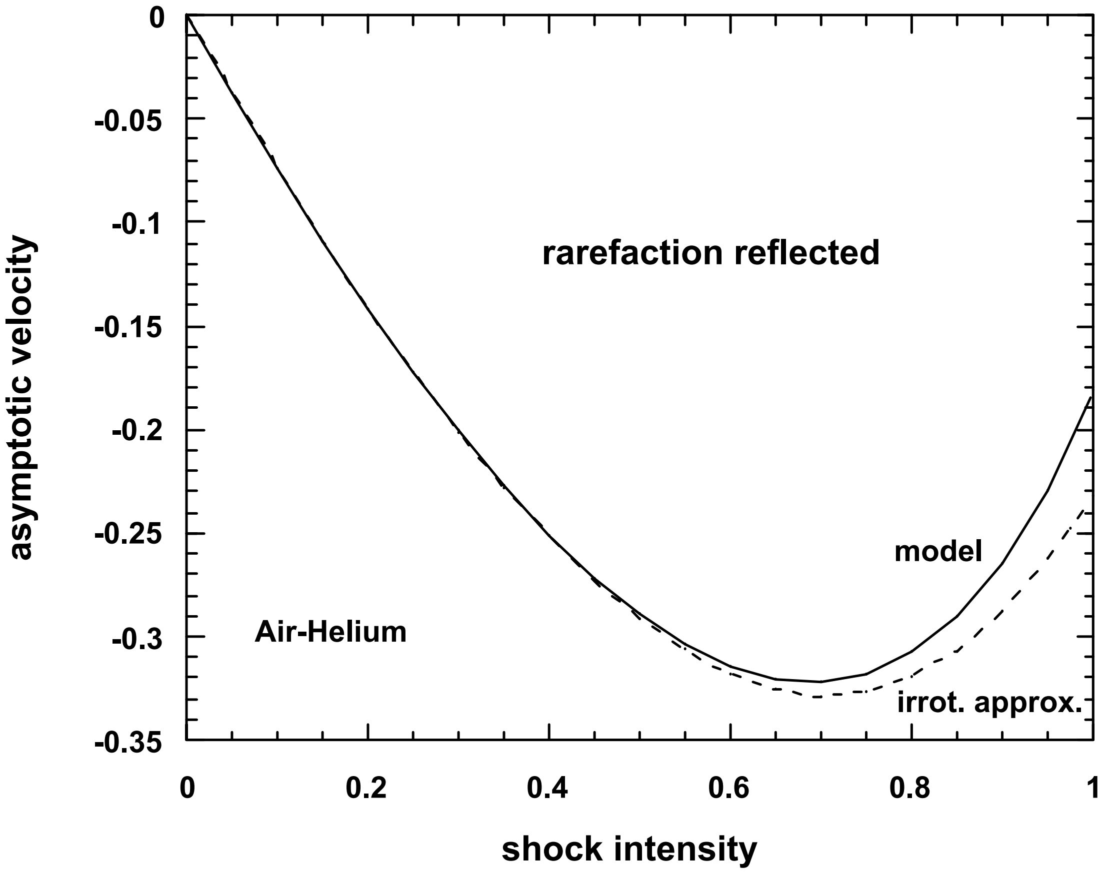
γlight = 1.3, γheavy = 1.8,
initial Atwood = 3
incident Mach = 2.84
linear theory
model
impulsive model
contact surface velocity
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2


Asymptotic normal velocity profile (t >> λ/c sound)
A shock has been reflected
incident Mach = 2.84
γlight = 1.3
γheavy = 1.8
pre-shock Atwood = 0.5
δvx
model
impulsive model
0,12
0,08
0,04
0
0
0,2
0,4
0,6
0,8
1

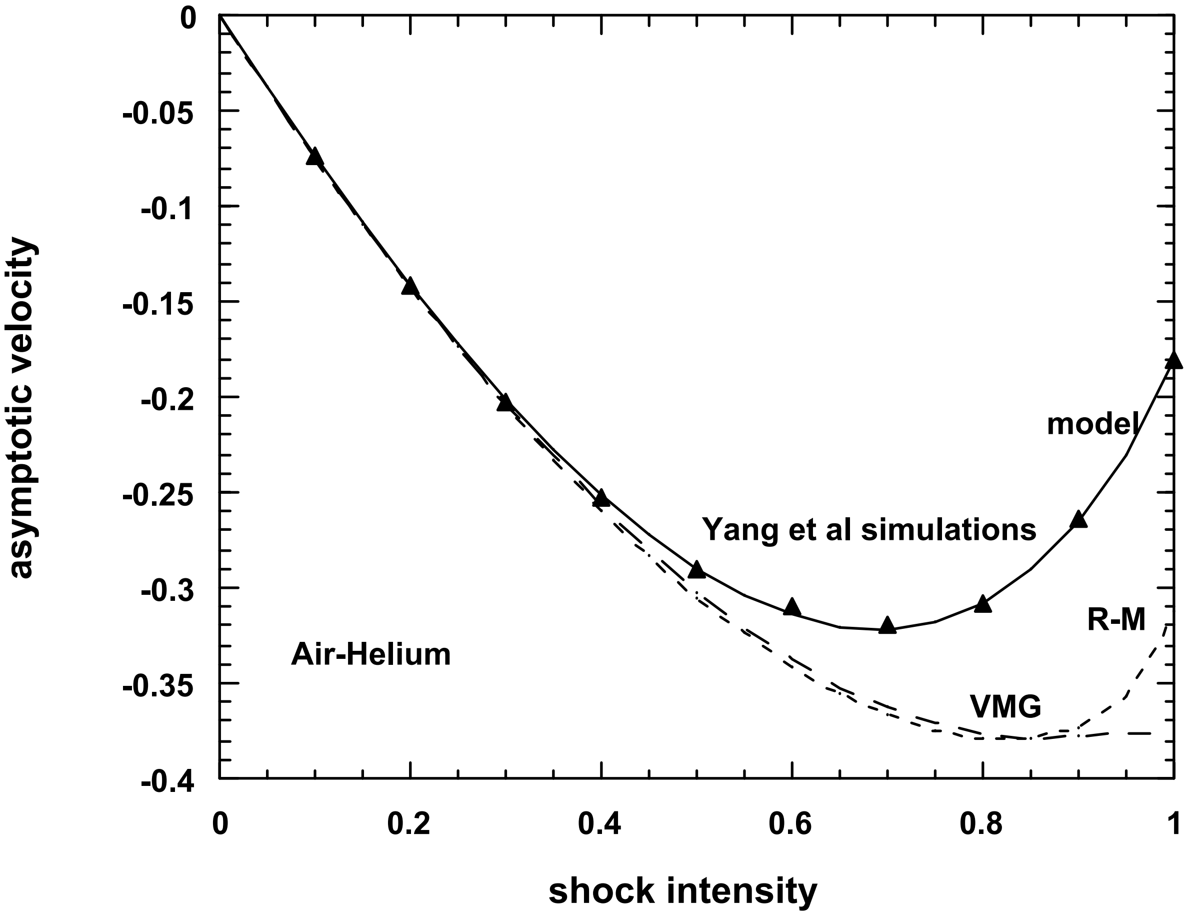
Comparison between the model and simulations together with two impulsive prescriptions. A shock has been reflected.

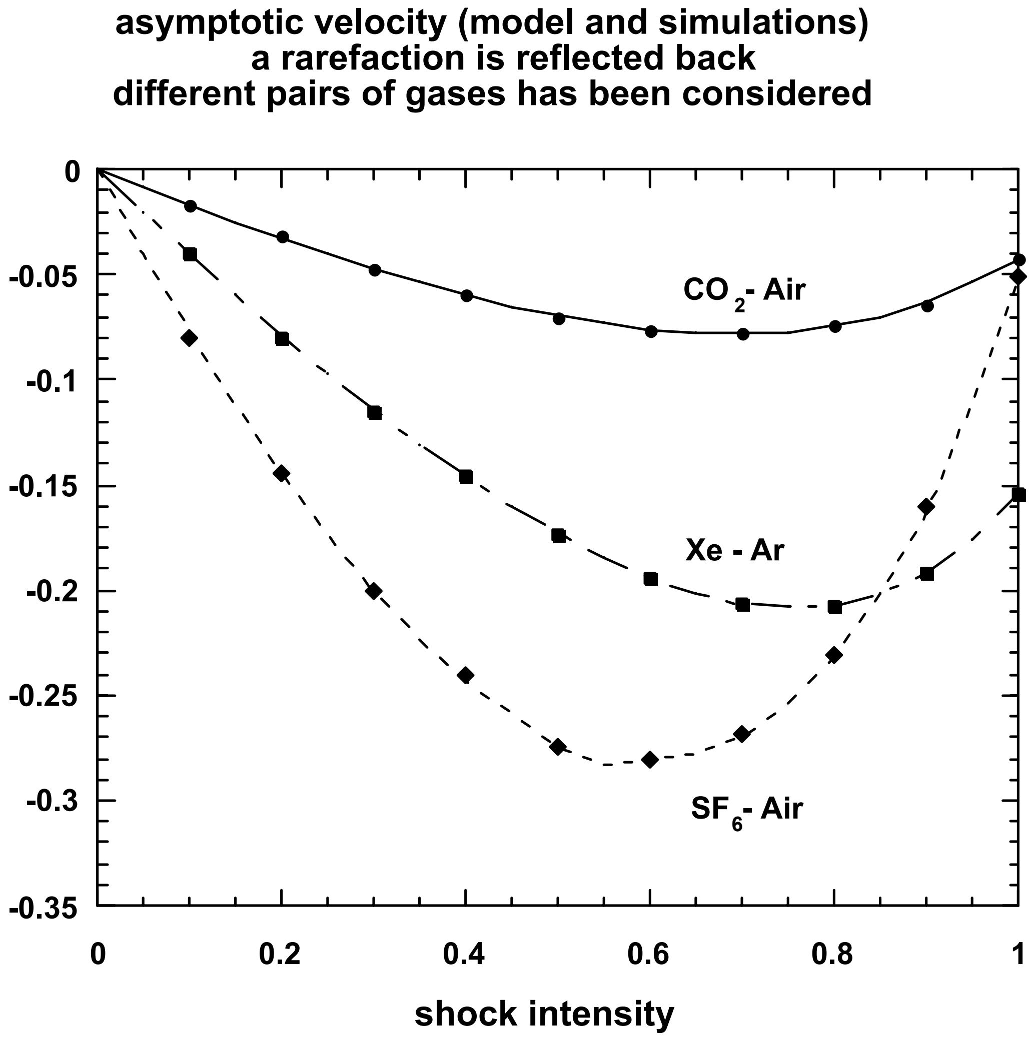


Comparison between the model and the irrotational approx. which does not include the vorticity corrections.
A shock impinged upon a Air-He interface



The same as the former picture but with a comparison to different impulsive prescriptions.





asymptotic velocity (model and simulations)
a rarefaction is reflected back
different pairs of gases has been considered
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
0
0.2
0.4
0.6
0.8
1
shock intensity
CO 2- Air
Xe - Ar
SF 6- Air