Combined Shear
and
Buoyancy Instabilities

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Spectral Turbulence Model for Variable Density Turbulence

- Steinkamp, Clark and Harlow, Int. J. of Multiphase Flow (1999)

- $$\hat{R}_{ij}(\vec{x},\vec{k},t) = \iiint\limits_{-\infty \text{ to } \infty} R_{ij}(\vec{x}_1,\vec{x}_2,t)e^{-i\vec{k}\vec{r}}d\vec{r}$$

  where $\vec{x} = \dfrac{\vec{x}_1 + \vec{x}_2}{2}, \vec{r} = \vec{x}_2 - \vec{x}_1$

- $$R_{ij}(\vec{x},k,t) = \int\limits_{\Omega_k} \hat{R}_{ij}(\vec{x},\vec{k},t)\frac{k^2 d\Omega_k}{(2\pi)^3}$$

- $$R_{ij}(\vec{x},t) = \int_0^\infty R_{ij}(\vec{x},k,t)dk$$

Advantages of Spectral Transport Models Over Single-Point Models (i.e., k-ε):

- More generality, such as in the case of non-equilibrium transients
- Does not require a length scale or dissipation equation
- Greater flexibility with modeling, such as with non-local interactions in both physical and wavenumber space

Disadvantages:

- Greater Complexity
- More computationally expensive
- More Flexibility!

Governing Equations

$$\frac{D\tilde{u}_i}{Dt} = -\frac{1}{\bar{\rho}}\left[\frac{\partial R_{in}}{\partial x_n} + \frac{\partial \bar{p}}{\partial x_i}\right] + \upsilon_m \frac{\partial^2 \bar{u}_i}{\partial x_n^2} + g_i$$

$$\tilde{u}_i = \frac{\overline{\rho u_i}}{\bar{\rho}},\ R_{in} = \overline{\rho u_i'' u''_n} \qquad \text{where } u_i = \tilde{u}_i + u_i''$$

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho}\tilde{u}_n}{\partial x_n} = 0$$

$$\frac{DR_{ij}(k)}{Dt} = \iiint_{-\infty \text{to} \infty} \left[a_i(k)\frac{\partial \bar{p}}{\partial x_j} + a_j(k)\frac{\partial \bar{p}}{\partial x_i}\right]_{\vec{x}'} \hat{f}(\vec{x}, \vec{x}')d\vec{x}' - R_{in}\frac{\partial \tilde{u}_j}{\partial x_n} - R_{jn}\frac{\partial \tilde{u}_i}{\partial x_n}$$

$$+ c_d \frac{\partial}{\partial x_n}\left[\upsilon_t \frac{\partial R_{ij}}{\partial x_n}\right] + c_m\left(\frac{1}{3}\delta_{ij}R_{nn} - R_{ij}\right)\int_0^\infty \sqrt{\frac{kR_{nn}}{\bar{\rho}}}dk$$

$$+ \frac{\partial}{\partial k}\left\{k^2\sqrt{\frac{kR_{nn}}{\bar{\rho}}}\left[-c_1 R_{ij} + c_2 k \frac{\partial R_{ij}}{\partial k}\right]\right\}$$

$$+ \text{Non-Local}(T(k))$$

where $a_i(\vec{x}_1, \vec{x}_2, t) = -\overline{u_i''(\vec{x}_1)\rho(\vec{x}_1)v(\vec{x}_2)}$ and $\upsilon_t = \int_0^\infty \sqrt{\frac{kR_{nn}}{\bar{\rho}}}\frac{dk}{k^2}$

Governing Equations (cont.)

$$\frac{Da_i(k)}{Dt} = \frac{b(k)}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} - \frac{R_{in}}{\overline{\rho}^2}\frac{\partial \overline{\rho}}{\partial x_n} + c_d \frac{\partial}{\partial x_n}\left[\upsilon_t \frac{\partial a_i}{\partial x_n}\right]$$

$$-\left[c_{r1} k^2 \sqrt{a_n a_n} + c_{r2} k \sqrt{\frac{k R_{nn}}{\overline{\rho}}}\right] a_i$$

$$+\frac{\partial}{\partial k}\left\{k^2 \sqrt{\frac{k R_{nn}}{\overline{\rho}}}\left[-c_1 a_i + c_2 k \frac{\partial a_i}{\partial k}\right]\right\}$$

where $b(\vec{x}_1, \vec{x}_2, t) = -\overline{\rho'(\vec{x}_1) v'(\vec{x}_2)}$

$$\frac{Db(k)}{Dt} = u_n \frac{\partial b}{\partial x_n} + \frac{2\overline{\rho} - \rho_1 - \rho_2}{\rho_1 \rho_2}\frac{\partial \overline{\rho} a_n}{\partial x_n} + c_d \frac{\partial}{\partial x_n}\left[\upsilon_t \frac{\partial b}{\partial x_n}\right]$$

$$-c_{fb}\left[\overline{v}^2 \frac{\partial \overline{\rho}/\overline{v}}{\partial x_n}\right]\frac{\partial k a_n}{\partial k} + \frac{\partial}{\partial k}\left\{k^2 \sqrt{\frac{k R_{nn}}{\overline{\rho}}}\left[-c_1 b + c_2 k \frac{\partial b}{\partial k}\right]\right\}$$

$$-2\upsilon_t k^2 b$$

Non-Local Behavior in Physical and Wavenumber Space

Physical Space:

$$\frac{DR_{nn}}{Dt} = 2\int_{-\infty}^{\infty} a_2(y')\frac{\partial \bar{p}}{\partial y'}\hat{f}(y,y')dy' + \cdots$$

$$\int_{-\infty}^{\infty}\hat{f}(y,y')dy = 1$$

$$\hat{f}(y,y') = \frac{\exp(-2k|y-y'|)}{\int_{-\infty}^{\infty}\exp(-2k|y-y''|)dy''}$$

Wavenumber Space: Kraichnan and Spiegal Model for Energy Transfer (1962)

$$T(k) = \int [S_a(k/p) - S_e(k/p)]dp$$

$$S_e(p/k) = \eta\sqrt{pE(p)}E(p)(\frac{p}{k})^m g(\frac{p}{k})$$

$$S_a(k/p) = S_e(p/k)$$

$$T(k) = -\Sigma(k) + \int \Sigma(p)\breve{f}(k,p)dp \quad \text{where} \quad \Sigma(k) = \eta[kE(k)]^{\frac{3}{2}}$$

$$\text{and} \quad \breve{f}(k,p) = p^{-1}(\frac{k}{p})^m g(\frac{k}{p})$$

Modeling Constants

c_1, c_2, c_m, c_{r1}, c_{r2}, c_d also η and $g(x)$

$c_1/c_2 = 2$ for equipartition of energy ($c_1 = .5$)

$c_{r1} \approx c_{r2} = 5.0$ to match buoyancy experiments

$c_m = 1.0$ original SCH model constant

Given that $\int \breve{f}(k,p)dk = 1 \Rightarrow \int x^m g(x)dx = 1$ where $x = k/p$

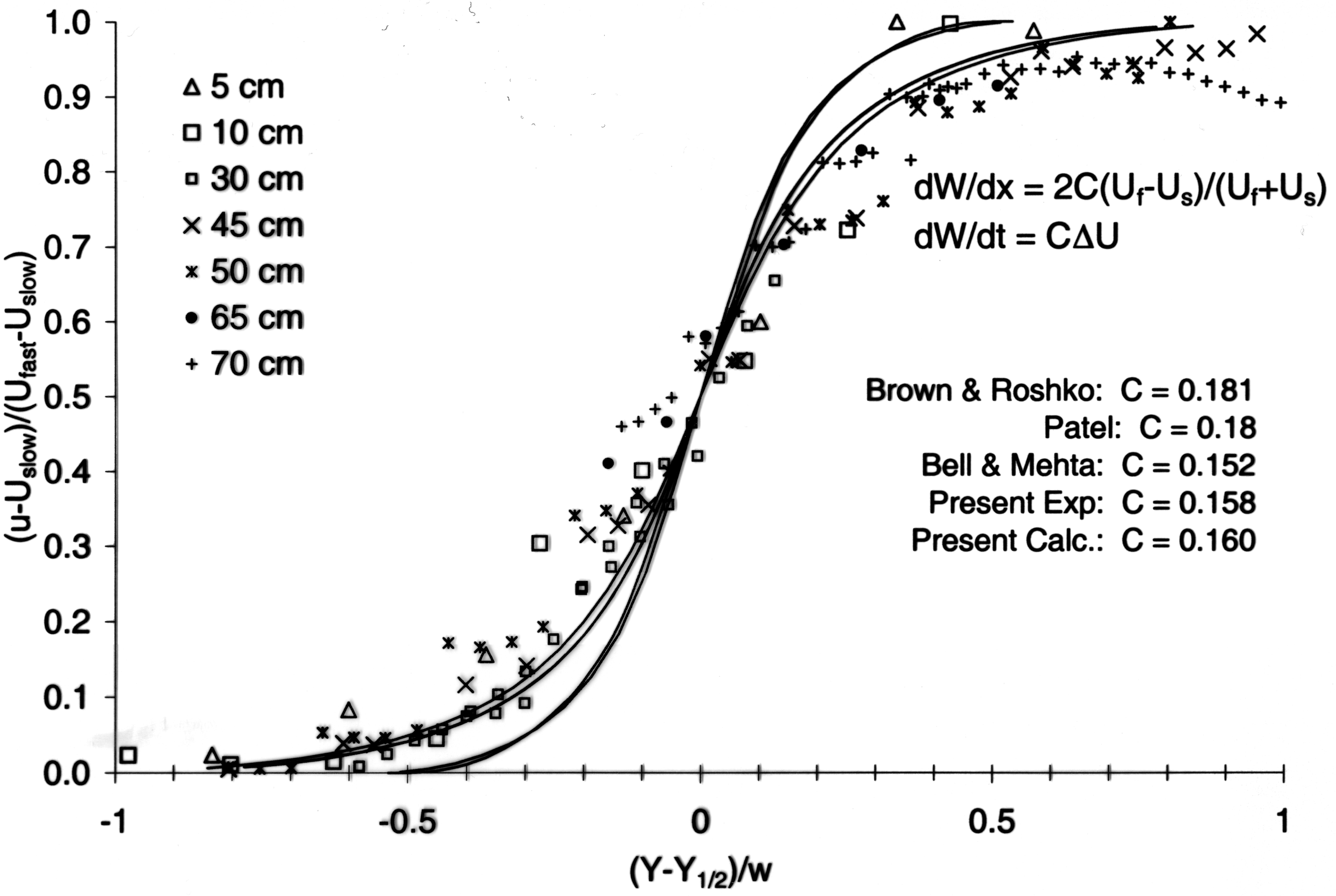
Let $g(x) = \exp\left[-\alpha(x + x^{-1})\right]/N$

and for 2k coupling ($m = -2, \alpha = 2/3$)

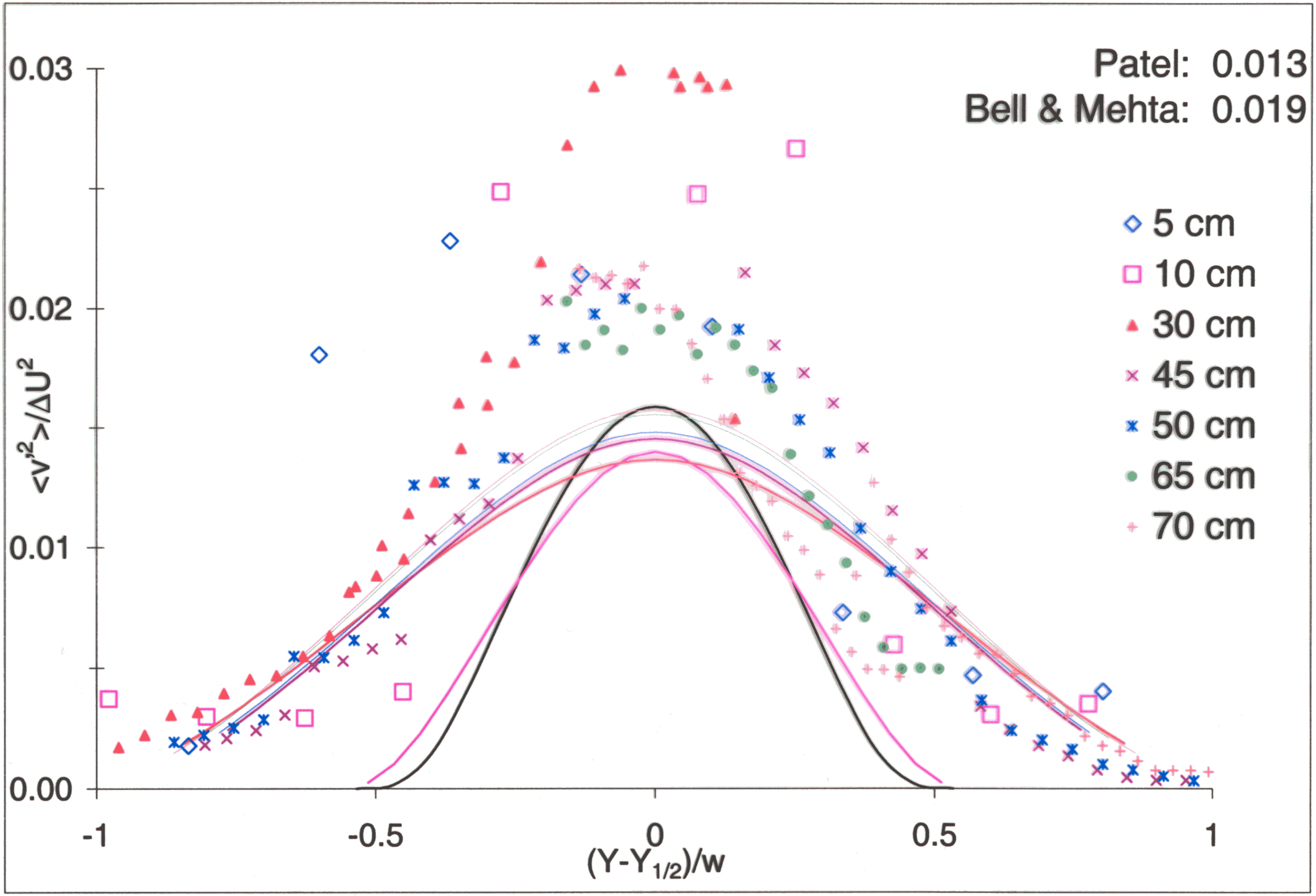
Lastly,

$$\varepsilon = \int_k^\infty \int_0^k S(q/p)dpdq$$

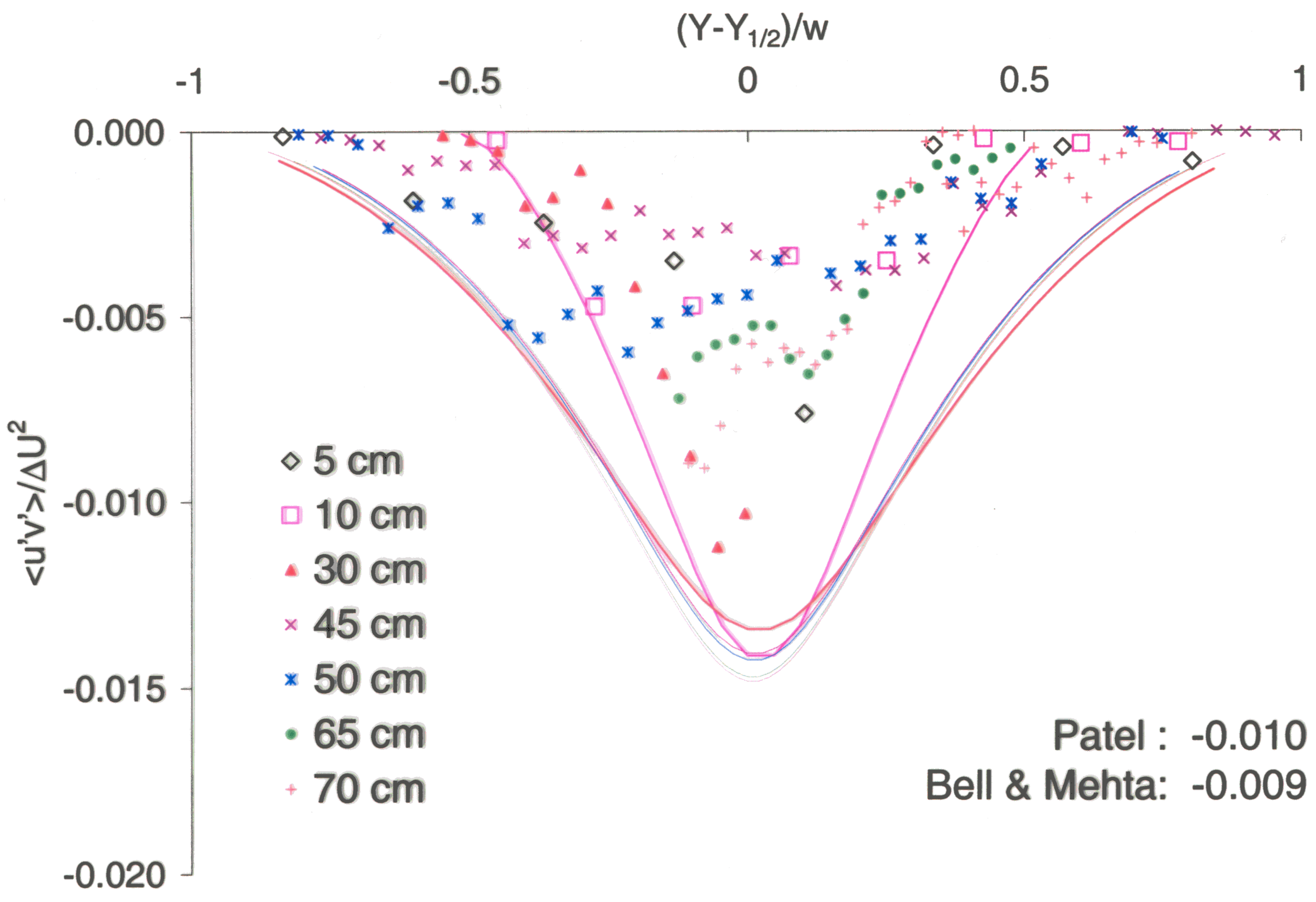
$$\eta = C_k^{-3/2}\left[\int_1^\infty \ln(x)(x^m - x^{-m-2})g(x)dx\right]^{-1} \Rightarrow \eta = 1.36$$



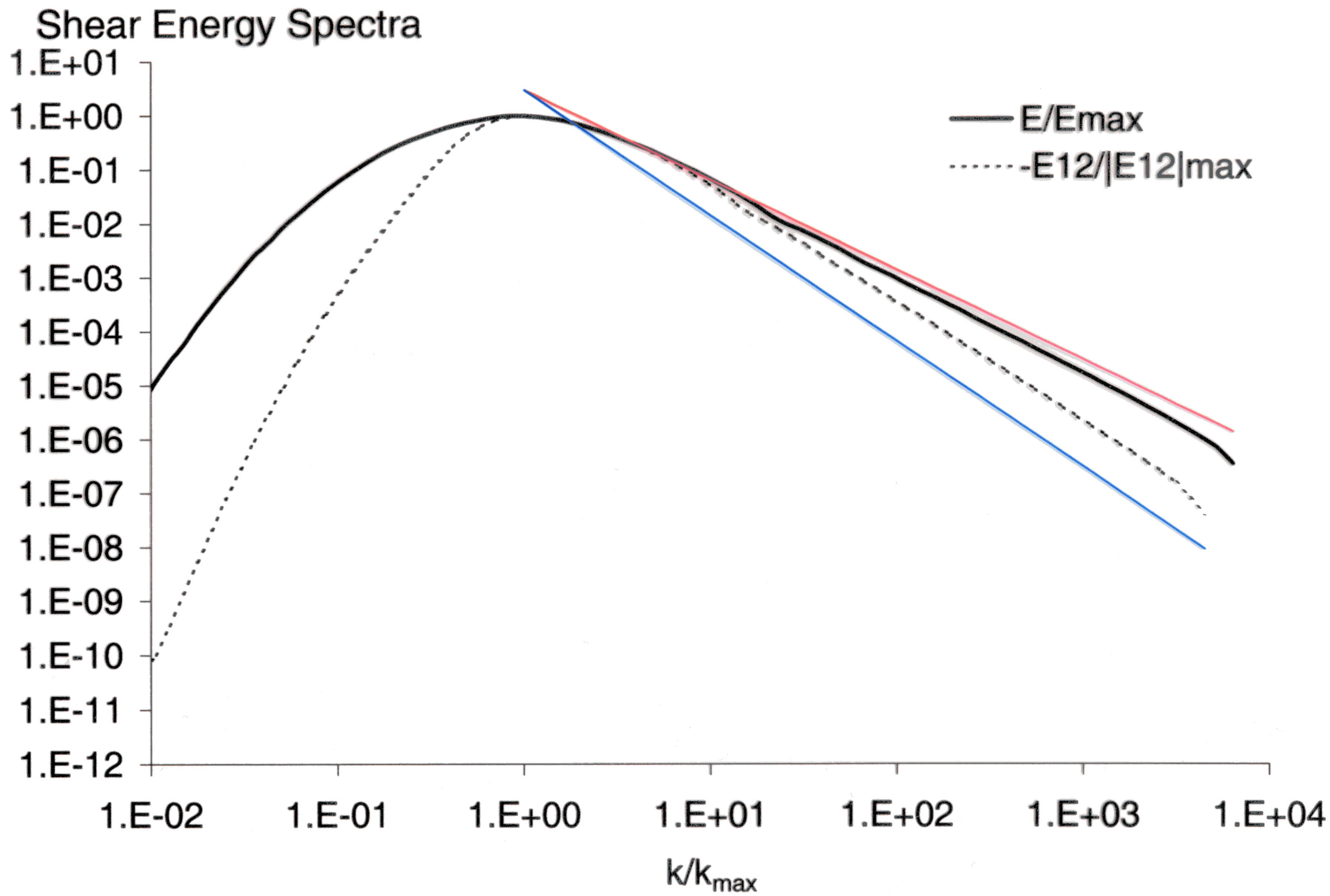
(u-U_{slow})/(U_{fast}-U_{slow})
$(Y-Y_{1/2})/w$
5 cm
10 cm
30 cm
45 cm
50 cm
65 cm
70 cm
$dW/dx = 2C(U_f-U_s)/(U_f+U_s)$
$dW/dt = C\Delta U$
Brown & Roshko: C = 0.181
Patel: C = 0.18
Bell & Mehta: C = 0.152
Present Exp: C = 0.158
Present Calc.: C = 0.160
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-1
-0.5
0
0.5
1

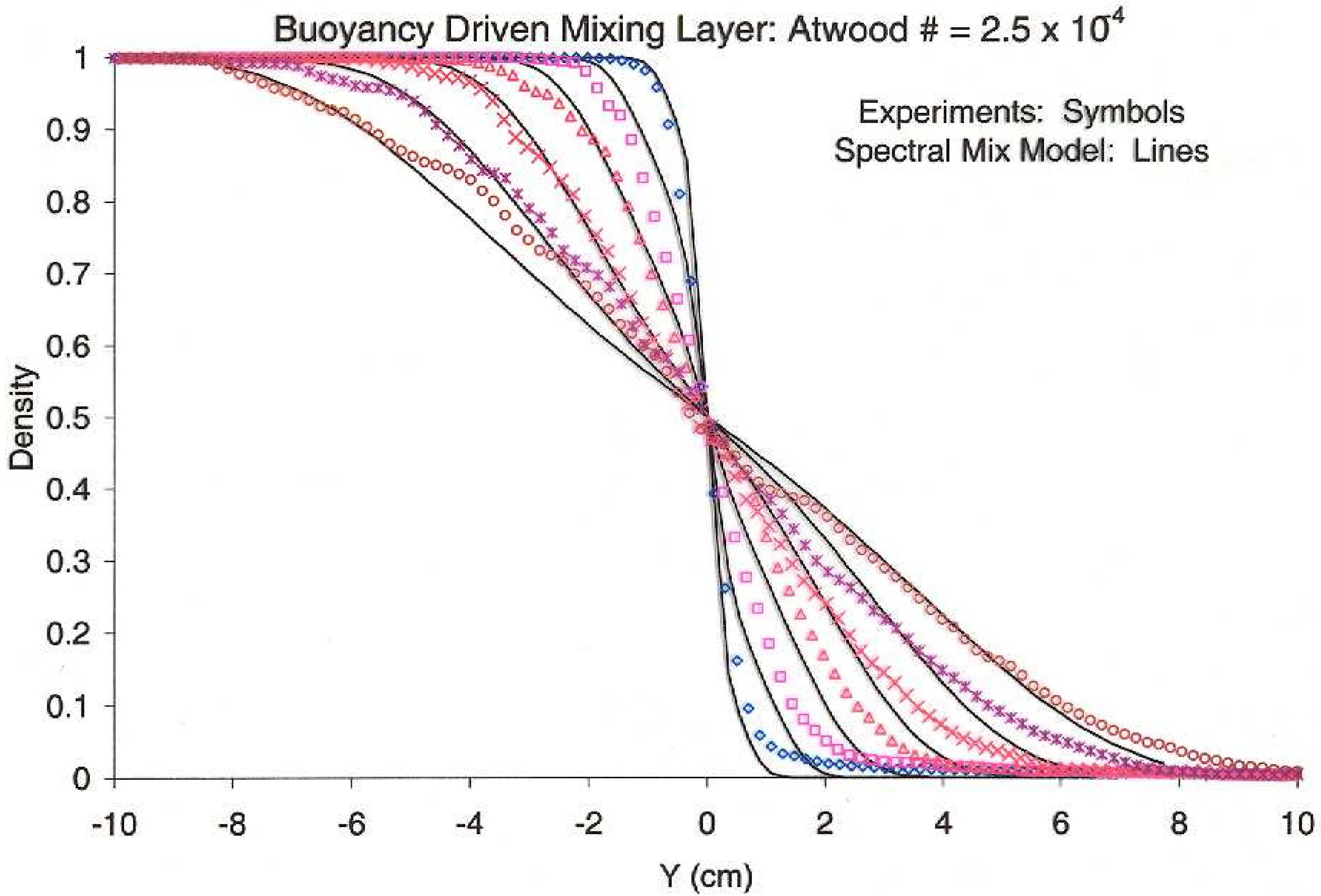


Patel: 0.013
Bell & Mehta: 0.019
5 cm
10 cm
30 cm
45 cm
50 cm
65 cm
70 cm
<v'²>/ΔU²
0.03
0.02
0.01
0.00
-1
-0.5
0
0.5
1
(Y-Y1/2)/w

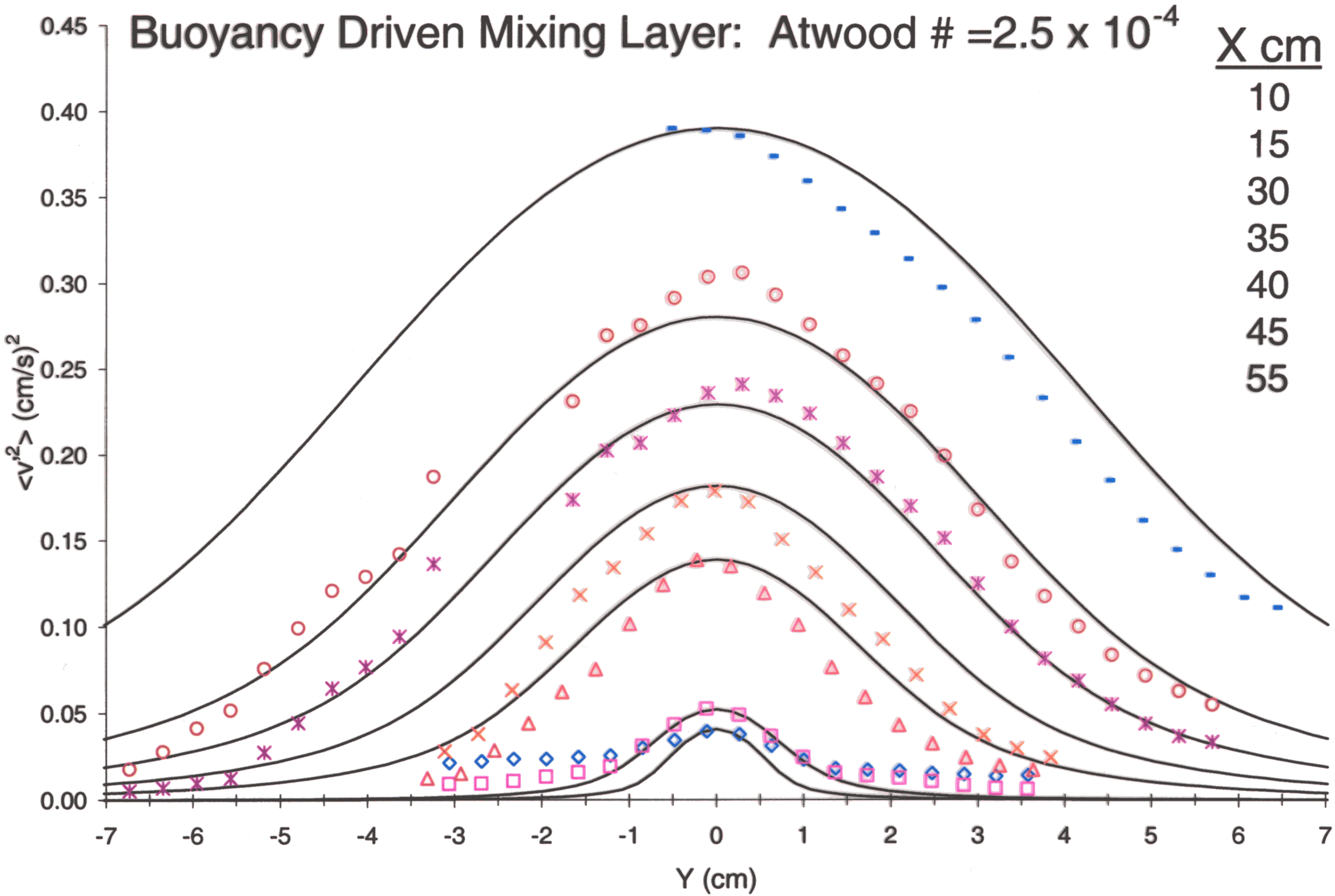


(Y-Y1/2)/w
-1
-0.5
0
0.5
1
0.000
-0.005
-0.010
-0.015
-0.020
<u'v'>/ΔU2
5 cm
10 cm
30 cm
45 cm
50 cm
65 cm
70 cm
Patel : -0.010
Bell & Mehta: -0.009

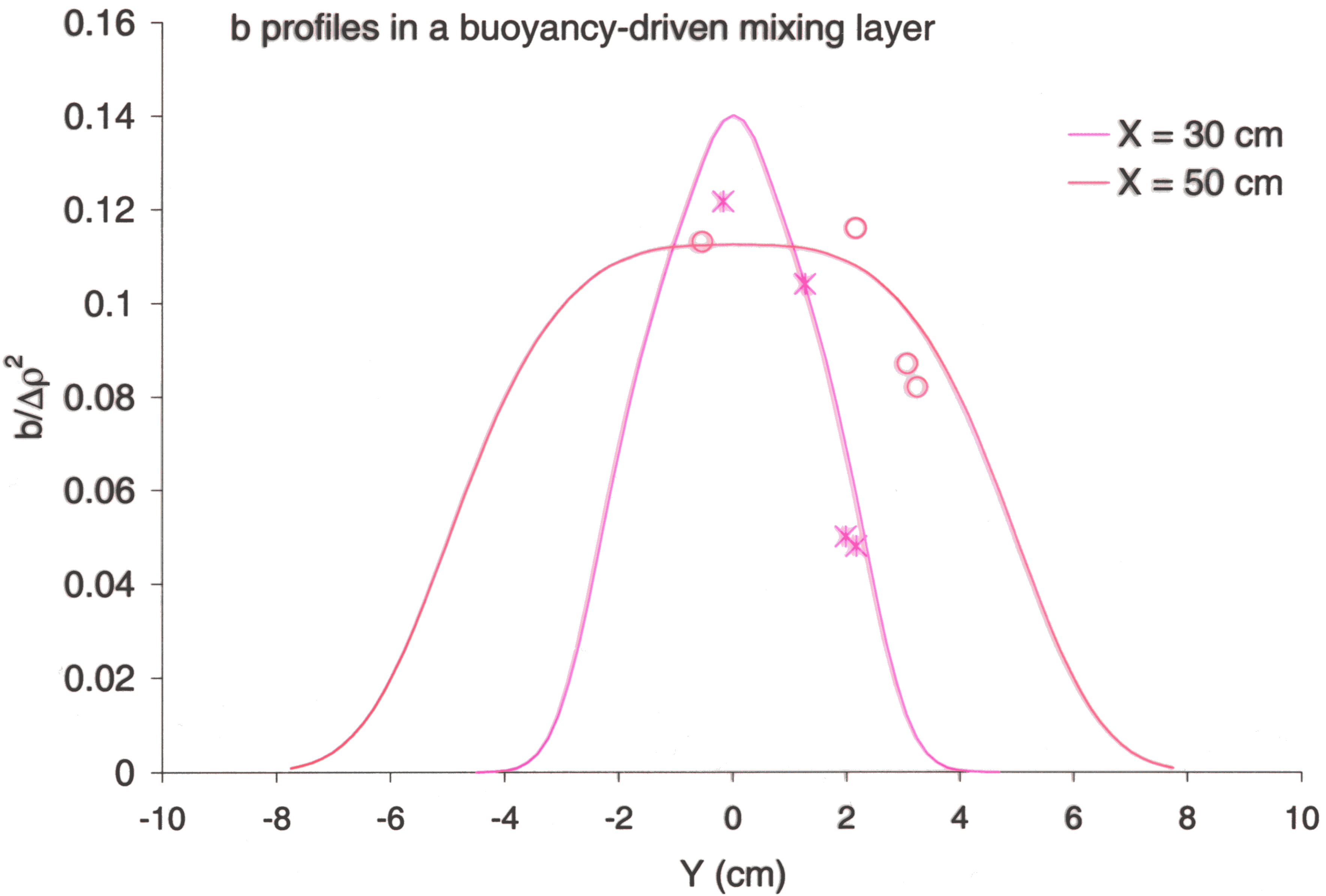

Shear Energy Spectra
1.E+01
1.E+00
1.E-01
1.E-02
1.E-03
1.E-04
1.E-05
1.E-06
1.E-07
1.E-08
1.E-09
1.E-10
1.E-11
1.E-12
E/Emax
-E12/|E12|max
1.E-02
1.E-01
1.E+00
1.E+01
1.E+02
1.E+03
1.E+04
k/k_{max}



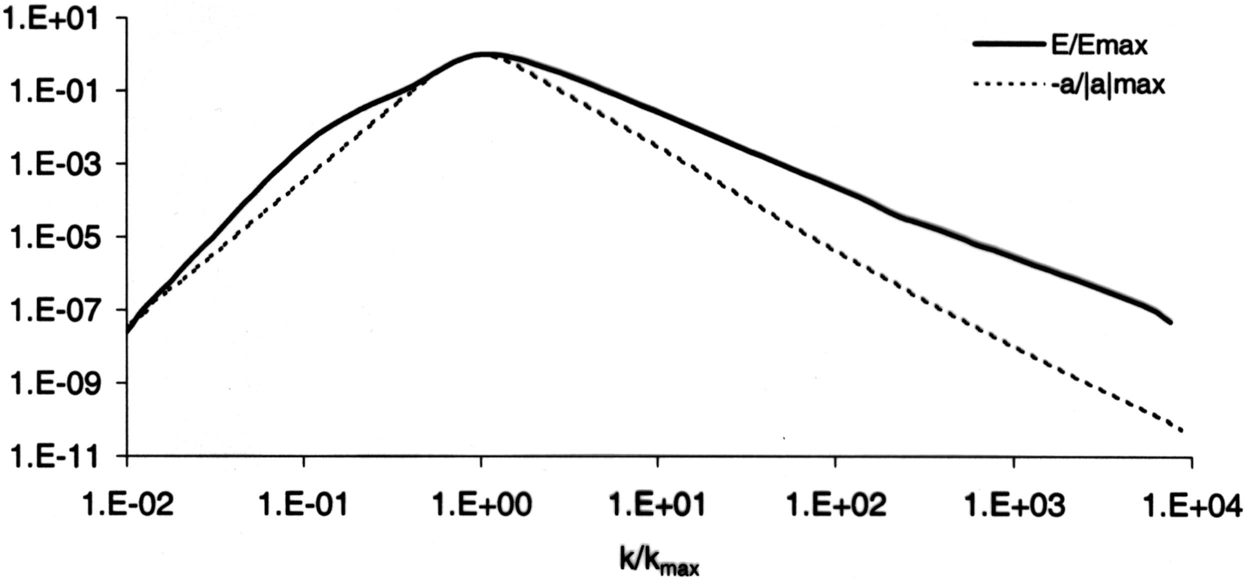
Buoyancy Driven Mixing Layer: Atwood # = 2.5 x 10-4
Experiments: Symbols
Spectral Mix Model: Lines
Density
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-8
-6
-4
-2
0
2
4
6
8
10
Y (cm)

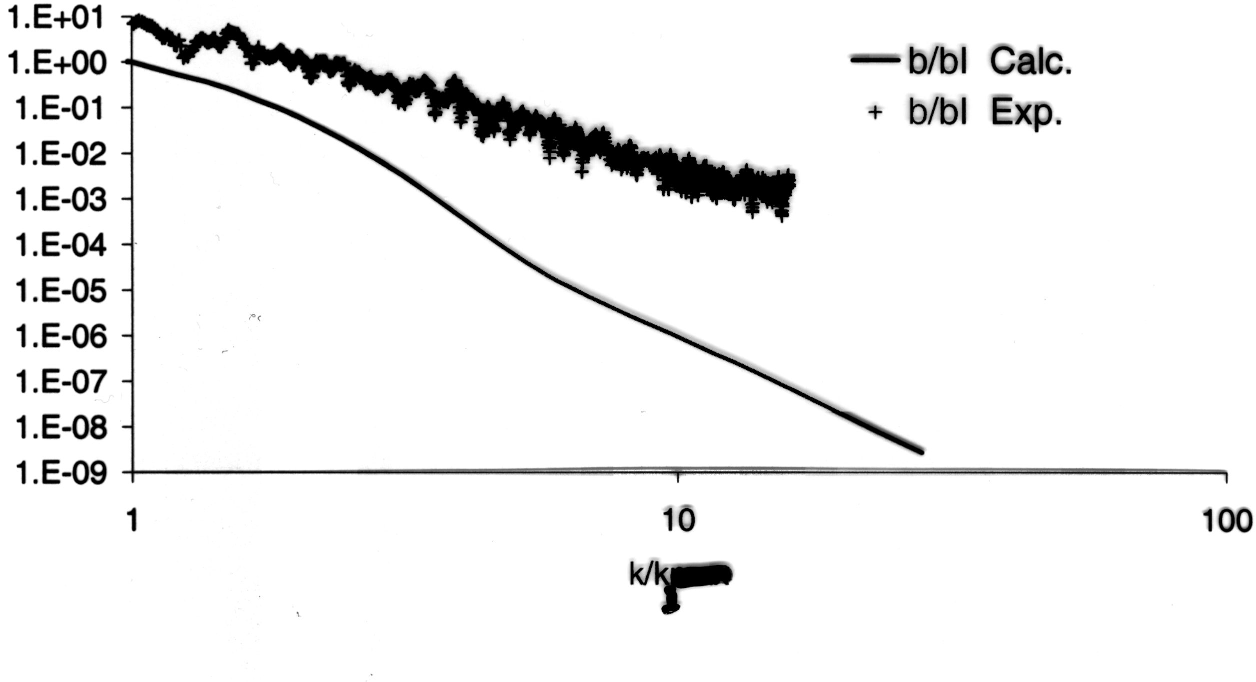


Buoyancy Driven Mixing Layer: Atwood # =2.5 x 10^-4
X cm
10
15
30
35
40
45
55
<v'^2> (cm/s)^2
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
Y (cm)



b profiles in a buoyancy-driven mixing layer
X = 30 cm
X = 50 cm
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
b/Δρ²
-10
-8
-6
-4
-2
0
2
4
6
8
10
Y (cm)


E/Emax
-a/|a|max
1.E+01
1.E-01
1.E-03
1.E-05
1.E-07
1.E-09
1.E-11
1.E-02
1.E-01
1.E+00
1.E+01
1.E+02
1.E+03
1.E+04
k/k_{max}


b/bl Calc.
b/bl Exp.
1.E+01
1.E+00
1.E-01
1.E-02
1.E-03
1.E-04
1.E-05
1.E-06
1.E-07
1.E-08
1.E-09
1
10
100
k/k[redacted]