Efficient perturbation methods for Richtmyer-Meshkov and Rayleigh-Taylor instabilities: Weakly nonlinear stage and Beyond.

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- I- Perturbation methods for the Richtmyer-Meshkov instability

  Summary, new approach and applications

- II- Beyond the weakly nonlinear stage

  Inferences and numerical tests

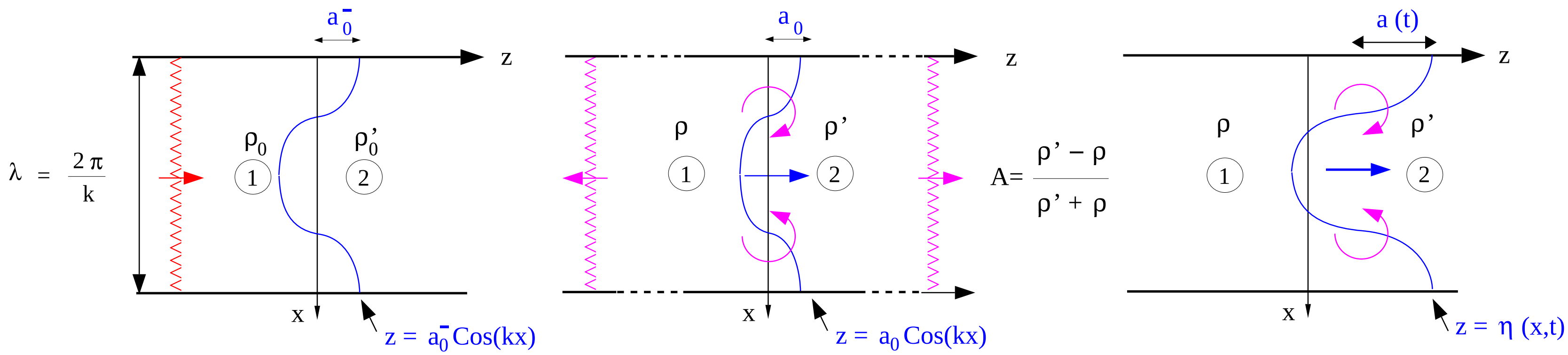
- III- Accurate approximation for the Rayleigh-Taylor instability

  Some examples

- Concluding remarks

I-Perturbation methods for the Richtmyer-Meshkov instability

Principle and hypothesis



- Fluids are considered **incompressible** and **irrotational**

- No more effect of the reflected and transmitted waves

Equations

- Velocities derive from potentials: $\vec{v} = -\vec{Grad}\ \Phi \qquad \vec{v'} = -\vec{Grad}\ \Phi'$

- Motion of the interface: $-\dfrac{\partial \Phi}{\partial z} = \dfrac{\partial \eta}{\partial t} - \dfrac{\partial \Phi}{\partial x}\dfrac{\partial \eta}{\partial x}$ and $-\dfrac{\partial \Phi'}{\partial z} = \dfrac{\partial \eta}{\partial t} - \dfrac{\partial \Phi'}{\partial x}\dfrac{\partial \eta}{\partial x}$ at $z = \eta$

- Bernoulli's equation: $-\rho'\dfrac{\partial \Phi'}{\partial t} + \rho\dfrac{\partial \Phi}{\partial t} + \dfrac{1}{2}\rho'[(\dfrac{\partial \Phi'}{\partial x})^2 + (\dfrac{\partial \Phi'}{\partial z})^2] - \dfrac{1}{2}\rho[(\dfrac{\partial \Phi}{\partial x})^2 + (\dfrac{\partial \Phi}{\partial z})^2] = 0$ at $z = \eta$

I-Perturbation methods for the Richtmyer-Meshkov instability

Summary of the full perturbation theory

- Find a small parameter: $a_0 k << 1$
- Expand unknowns η, Φ, Φ' as series: $\eta = \sum \eta^{(n)}$ with $\eta^{(n)} \# (a_0 k)^n$
- Introduce expansions in the 3-equation system
- Collect terms of the same order
- Solve resulting systems $\Longrightarrow$ appearance of "secular terms"

  Ex: $$X^{(n)} = (a_0 k)^n \times \left(x_1^{(n)} t^n + x_2^{(n)} t^{n-1} + ... + x_n^{(n)} \right)$$

- At the n^{th} order, $3n^2$ coefficients must be evaluated

$$k\eta^{(3)} = -\frac{1}{24}(a_0 k)^3[(4A^2+1)(\sigma t)^3 + 3(\sigma t)^2 + 6\sigma t]\ coskx + \frac{1}{8}(a_0 k)^3[(4A^2-1)(\sigma t)^3 - 3(\sigma t)^2]\ cos3kx$$

I-Perturbation methods for the Richtmyer-Meshkov instability

How to simplify the full perturbation theory

C.M. Bender, S.A. Orszag, "Advanced mathematical methods ... ", 1978

At each order, keep ONLY the MOST SECULAR TERMS

- A new small parameter: $a_0 k\sigma t << 1$
- Expand unknowns η, Φ, Φ' as series: $\eta = \sum \eta^{(n)}$ with $\eta^{(n)} \# (a_0 k\sigma t)^n$
- Introduce expansions in the 3-equation system
- Collect terms of the same order $\Longleftrightarrow$ at the n^{th} order, $\frac{\partial \eta}{\partial t} \Rightarrow t^{n-1}$ terms
- Solve resulting systems: results are written in a new form

Ex: $$X^{(n)} = (a_0 k)^n \times \left(x_1^{(n)} t^n + x_2^{(n)} t^{n-1} + ... + x_n^{(n)}\right)$$
$$\Downarrow$$
$$X^{(n)} \approx (a_0 k)^n \times x_1^{(n)} t^n$$

- At the n^{th} order, only $3n$ coefficients must be evaluated

$$k\eta^{(3)} = -\frac{1}{24}(a_0 k)^3[(4A^2+1)(\sigma t)^3 + 3(\sigma t)^2 + 6\sigma t]\ cos kx + \frac{1}{8}(a_0 k)^3[(4A^2-1)(\sigma t)^3 - 3(\sigma t)^2]\ cos 3kx$$
$$\Downarrow$$
$$k\eta^{(3)} \approx (a_0 k\sigma t)^3 \left[-\frac{1}{24}(4A^2-1) cos\ kx + \frac{1}{8}(4A^2-1) cos\ 3kx\right]$$

- Same algorithm for multimode configuration

I-Perturbation methods for the Richtmyer-Meshkov instability

Some remarks about the approximate perturbation theory

- The approximate pertubation theory gives accurate results within its range of validity
- The range of validity is clearly defined: $a_0 k \sigma t_v = 1$
- The approximation makes solution easier and faster: n^2-algorithm $\rightarrow$ n-algorithm
- The approximation makes multimode configurations tractable
- Sign of the initial amplitudes (or the phase between modes) have effects upon the mode-competition process

$$\Downarrow$$

Mode-competition models must take into account the phases of the initial perturbation spectrum

Comparisons approximate theory/simulations
Single mode dynamics

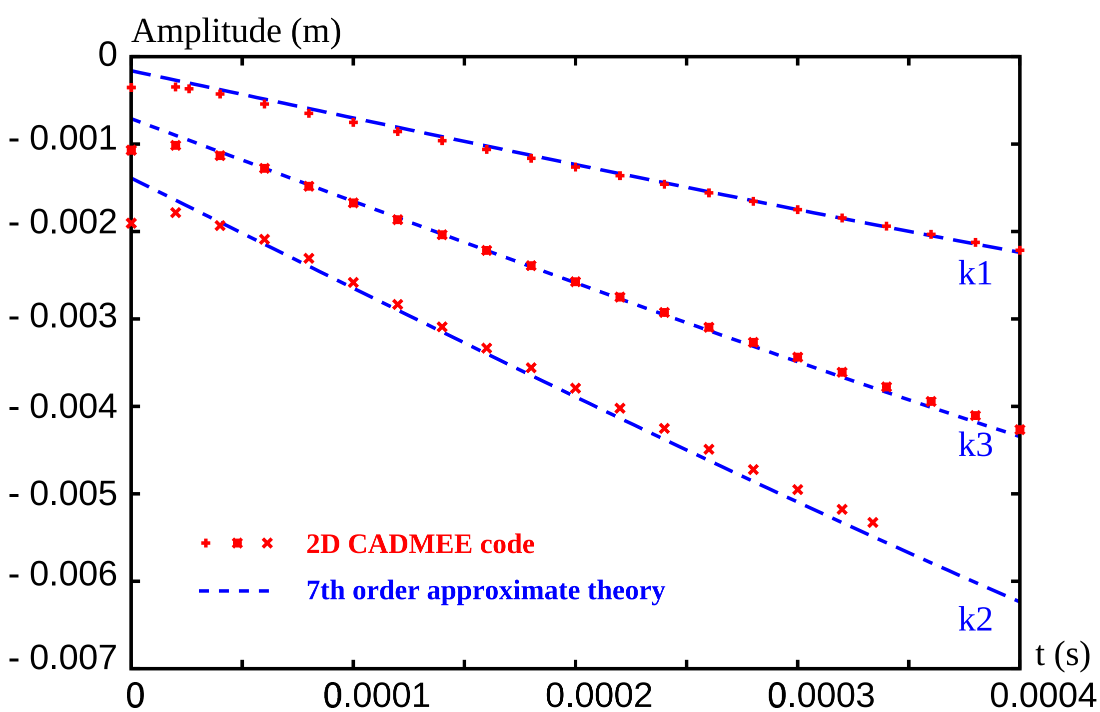
3 modes are considered: $k_1 = 274.855 \quad k_2 = 3/7\ k1 \quad k_3 = 4/7\ k1$

$$a_{01}^- = -0.35\ 10^{-3} m \quad a_{02}^- = -1.9055\ 10^{-3} m \quad a_{03}^- = -1.072\ 10^{-3} m$$

Physical parameters: He/Air $\quad M_0 = 1.0962 \quad A^- = 0.75724$

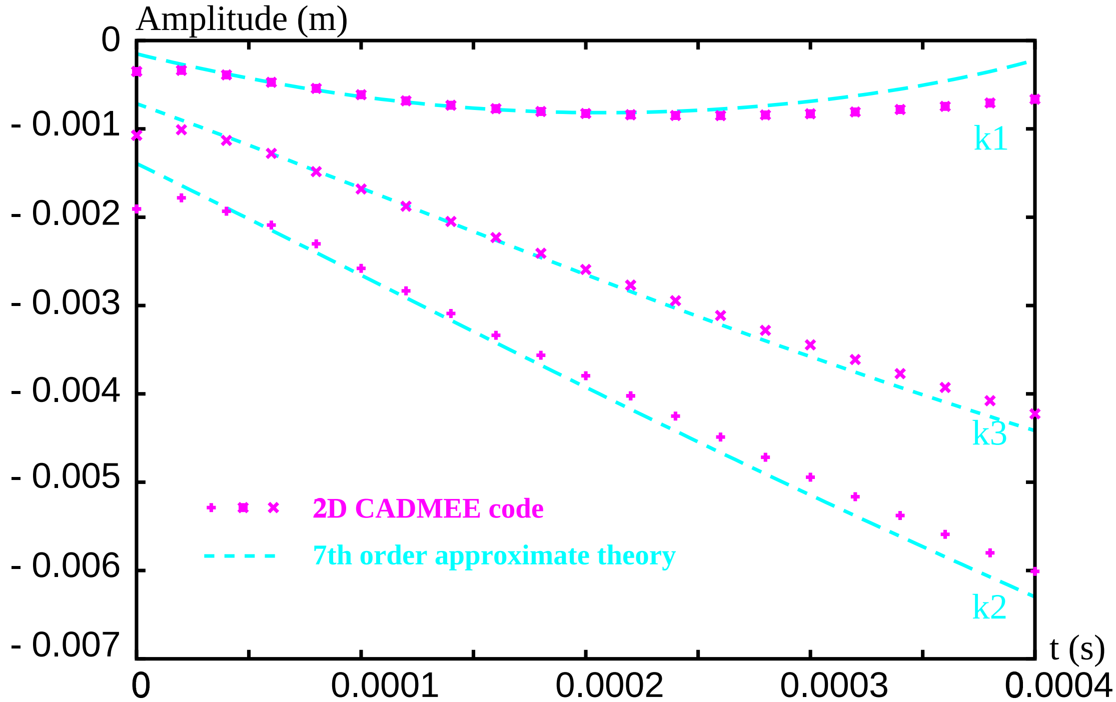
$$t_v = 0.67 ms$$

Simulations: 2D CADMEE code: 2^{nd} order in time and space



Multimode dynamics: negative initial amplitude

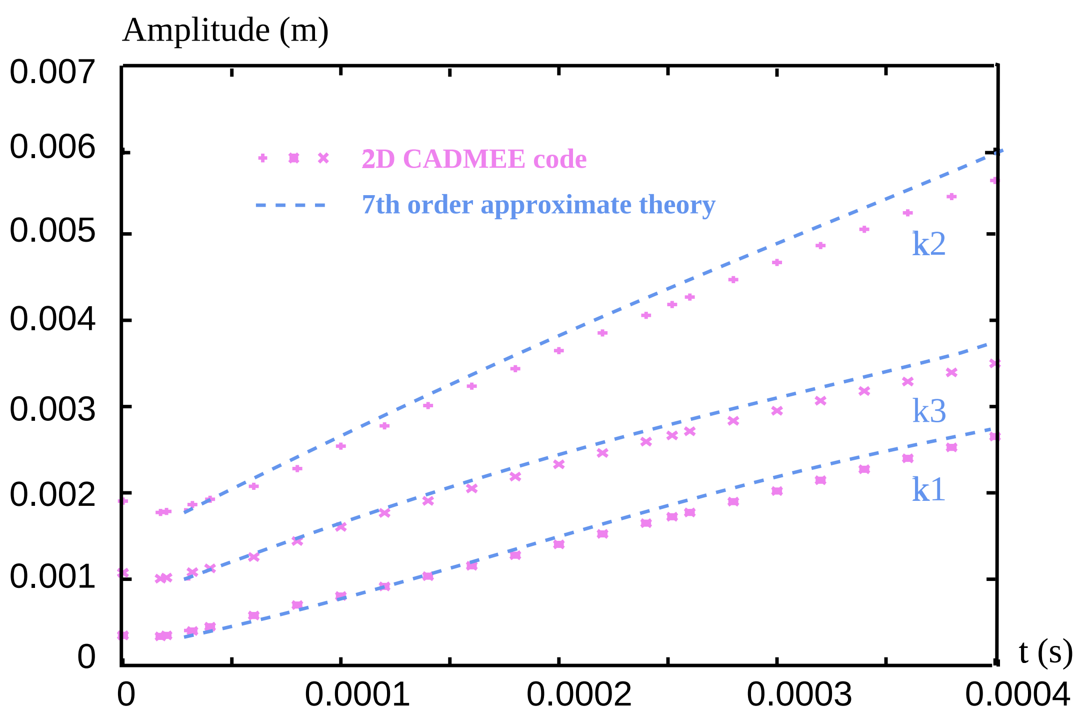
$a_{01}^- = -0.35\ 10^{-3} m \quad a_{02}^- = -1.9055\ 10^{-3} m \quad a_{03}^- = -1.072\ 10^{-3} m$



The growth of the mode k_1 is strongly reduced by k_2 and k_3.

Multimode dynamics: positive initial amplitude

$a_{01}^- = +0.35\ 10^{-3} m \quad a_{02}^- = +1.9055\ 10^{-3} m \quad a_{03}^- = +1.072\ 10^{-3} m$



The growth of the mode k_1 is unaffected by k_2 and k_3.

A-Late time 2-mode competition:

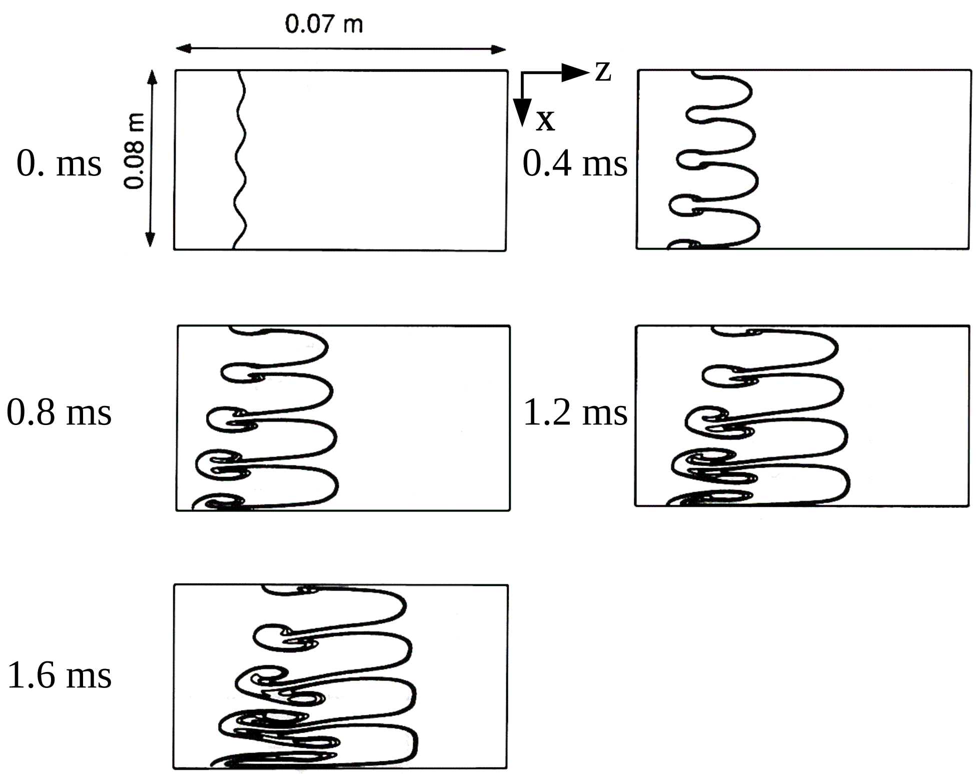
influence of the values of the initial amplitude

2 nearby modes are studied: $k_1 = 274.855 \quad k_2 = 8/7 k_1 = 314.159$

2 initial amplitudes are used: $a^-_{01} = 10^{-3}$ or $a^-_{02} = 0.35\ 10^{-3} m$

Physical parameters: He/Air $\quad M_0 = 1.26 \quad A^- = 0.756$

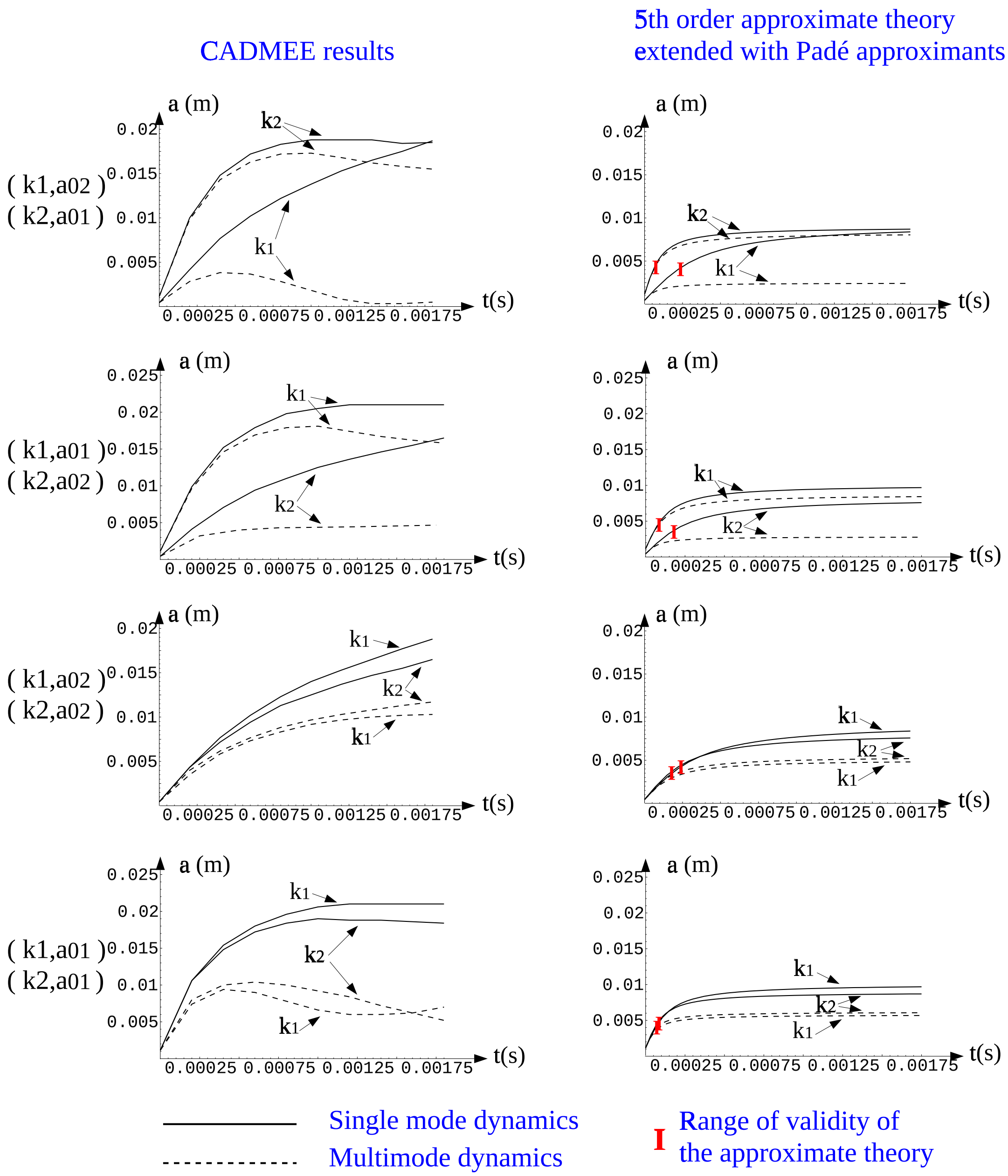
CADMEE results for configuration $(k_1, a^-_{02}), (k_2, a^-_{01})$-$t_v = 0.24\ ms$



A spectral analysis is performed:

- z-integration of the density: $\rho(x) = \int \rho(z,x) dz$
- Fourier transform of $\rho(x)$

Growth of the 2 dominant modes



- The values of the initial amplitudes dictate the selection process:
  - the mode with the lower initial amplitude is suppressed
  - if the amplitude are the same, both modes are equally lowered
- Qualitative agreement between simulation and theory: behavior and selection process are correctly predicted
- Quantitative discrepancies: Padé approximants underestimate the late-time nonlinear growth
- Late-time nonlinear behaviors seems settled since the early nonlinear stage

B-Class of homothetic single-mode perturbations

Perturbation expansion can be considered as Taylor expansion of an unknown function $\mathcal{F}$:

$$k\eta = \sum k\eta^{(n)} = \sum_{n=1}^{\infty} (a_0 k\sigma t)^n \sum_{j=1}^{n} a_j^{(n)} \, cos(jkx)$$

$$\Downarrow$$

$$k\,(a(T) - a_0) = \sum_{p=0}^{\infty} P_{2p+1}[A]\; T^{2p+1} = \sum_{p=0}^{\infty} \frac{1}{p!} \left.\frac{\partial^p \mathcal{F}}{\partial T^p}\right|_0 T^p = \mathcal{F}(A,T)$$

with $T = a_0 k \sigma t$

$$\Downarrow$$

If two perturbations 1 and 2 are such that:

$$A_1 = A_2 \text{ and } a_{01} k_1 \sigma_1 = a_{02} k_2 \sigma_2 \Longleftrightarrow a_{01} k_1^2 = a_{02} k_2^2$$

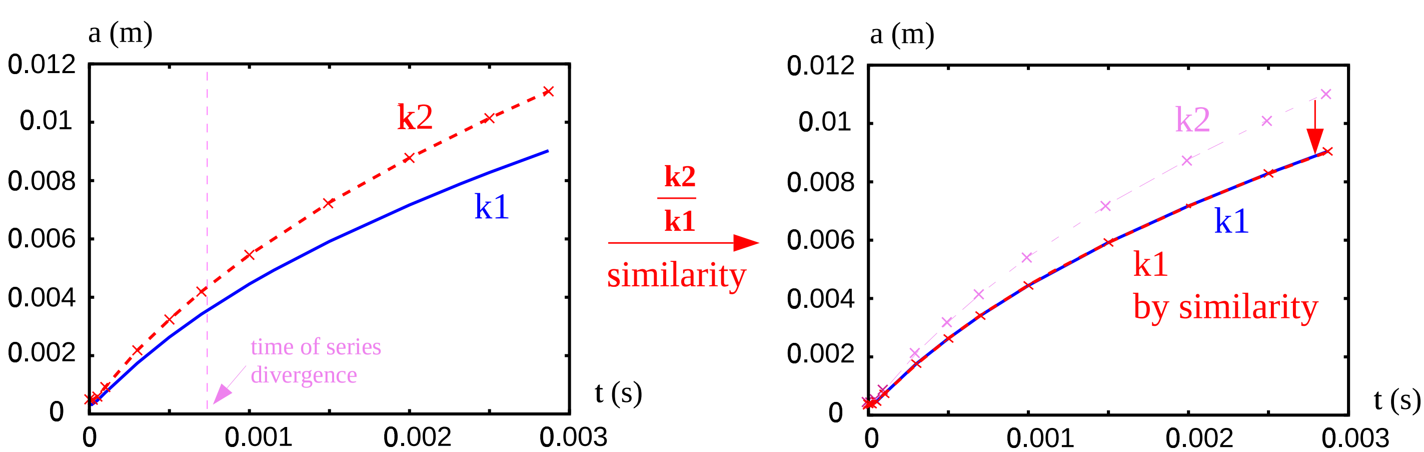
$$\text{Then } a_1(t) = \frac{k_2}{k_1}\,(a_2(t) - a_{02}) + a_{01}$$

Application to numerical simulations:

$$\text{He/Air} \quad M_0 = 1.26 \quad A^- = 0.756$$

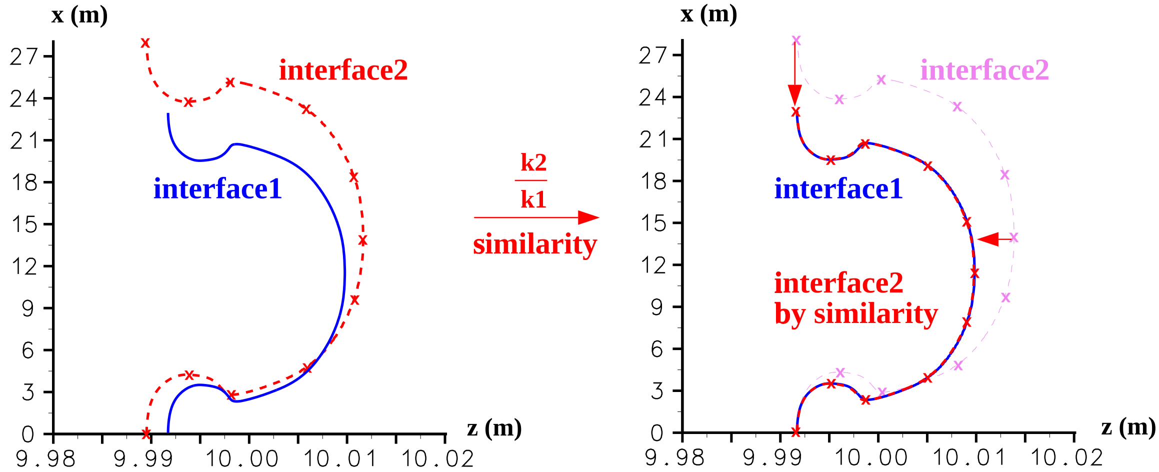
$$(a_{01}, k_1) = (0.35\ 10^{-3}, 274.855) \quad (a_{02}, k_2) = (0.525\ 10^{-3}, 224.418)$$

$$a_{01} k_1^2 = a_{02} k_2^2$$



The similarity still stands during the mushroom stage

$$a_{02} = 0.525mm > a_{01} = 0.35mm \qquad \lambda_2 = 2.8cm > \lambda_1 = 2.3cm$$



Some remarks

- A class of homothetic perturbated interfaces is inferred from the approximate perturbation theory:

$$A_i = A_j \qquad a_{0i} k_i^2 = a_{0j} k_j^2$$

- The theory is accurate
- The similarity stands during the late-time nonlinear stage: the shapes are homothetic even in the Kelvin-Helmholtz vortex area
- Large perturbations can be used to study small ones
  - Advantages for numerical simulations: less limiting CFL conditions (cpu time 2 = 78% cpu time 1)
  - Advantages for experiments: $(\lambda, a_0) = (10\mu m, 0.1\mu m) \Longleftrightarrow (31.62\mu m, 1\mu m) \Longleftrightarrow (1cm, 4mm)$

III- Accurate approximation for the Rayleigh-Taylor instability

- Same hypothesis for the fluids as for the RM instability
- Same 3-equation system as for the RM instability except for the additional

$(\rho' - \rho) g\eta$ term in Bernoulli's equation

- Same solution algorithm except that the "secular" terms write as

$$(a_0 k e^{\gamma t})^i \text{ with } \gamma = \sqrt{Agk}$$

- Unknowns are written in a similar form:

$$\eta = \sum \eta^{(n)} = \sum (a_0 k e^{\gamma t})^n \sum_{j=1}^{n} a_j^{(n)} cos(jkx)$$

- Range of validity: $a_0 k e^{\gamma t_v} = 1$
- Same advantage: n^2-algorithm $\rightarrow$ n-algorithm
- Example:

$$k\eta^{(2)} = -\frac{1}{4} A a_0^2 k^2 \left[\, 2\, Cosh(\sqrt{2}\gamma t) \; - 1 \; - Cosh(2\gamma t)\right] \cos 2kx$$

$$\downarrow$$

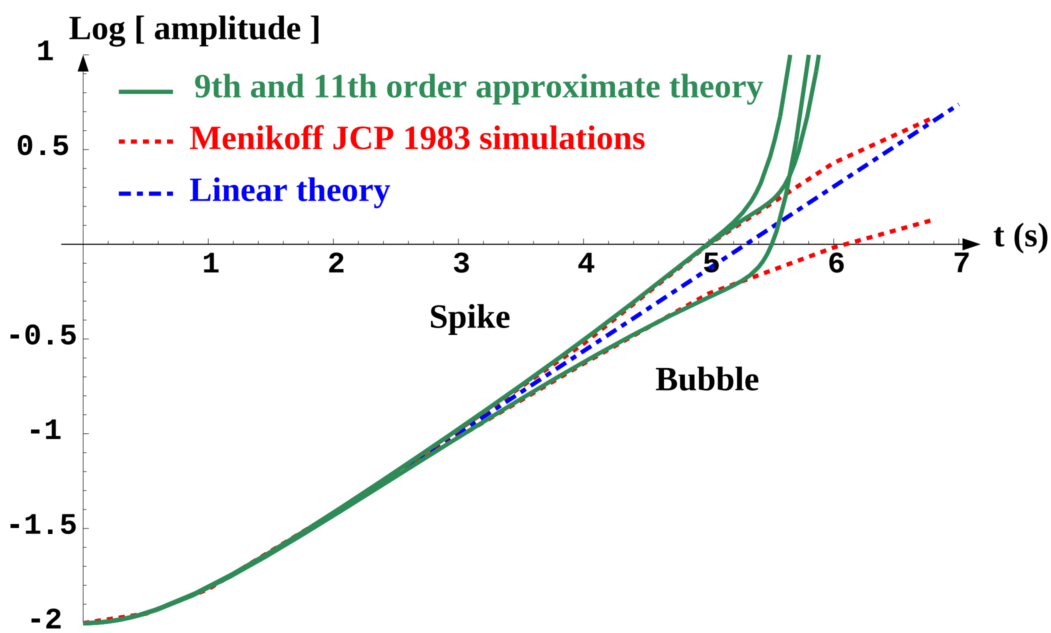
$$k\eta^{(2)} = -\frac{A}{8} (a_0 k)^2 e^{2\gamma t} \cos 2kx$$

Comparisons approximate theory / simulations

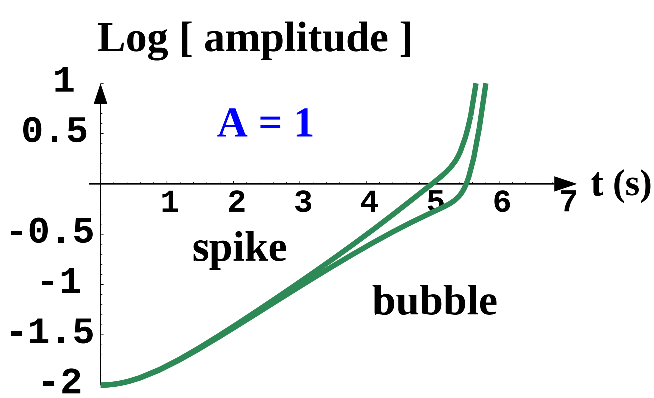
Trajectories of the tip of the bubble and the tip of the spike

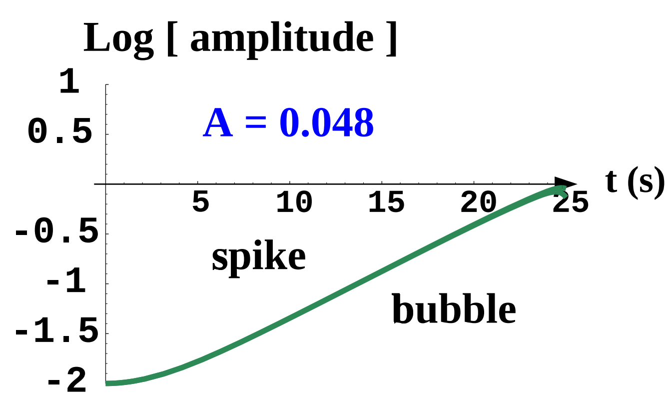
Menikoff et Zemach, JCP **51**, 28-64 (1983)

$A = 1;\ g = 1;\ k = 1;\ a_0 = -0.01 \Rightarrow t_v = 4.6$



Influence of the Atwood number:





- If $A \approx 1$, the spike grows faster than the bubble.
- If $A \approx 0$, bubble and spike have the same growth.
- Same conclusions as Baker *et al.*, Phys. Fluids **23**(8), 1980.

Concluding remarks

- Approximate perturbation theory is accurate within its range of validity for RM and RT instabilities
- Solutions are easier and faster: n^2-algorithm $\rightarrow$ n-algorithm
- For the Richtmyer-Meshkov instabilities:
  - Approximate theory makes multimode configurations tractable
  - Padé extended approximate theory allows us to predict late-time selection process: the origin of such process occurs in the weakly nonlinear stage
  - Mode spectrum, amplitude values but also, the phase between modes must be known to determine the modes leading to the bubble stage.
  - Existence of a class of homothetic perturbated interfaces:

$$A_i = A_j \quad \text{and} \quad a_{0i}\, k_i^2 = a_{0j}\, k_j^2$$