ANALYTICAL STUDY of the RTI in COMPRESSIBLE FLUIDS

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USE of COMPRESSIBILITY for the RTI

Sound speed: $c_s^2 = (\Delta p / \Delta \rho)_{adiab.}$

incompressible fluids: $\Delta \rho$ finite $\rightarrow \Delta p = \infty$!

$c_s \rightarrow \infty$

NOT NECESSARLY REPRESENTATIVE OF PHYSICS

- ICF
- ASTROPHYSICS

H-envelope Type II -SN progenitor:
$T \simeq 10^6$ K, $\rho \simeq 10^{-3}$ g/cc, $D \simeq c/10$ $M \simeq 100$'s!

INCOMPRESSIBILITY IS NO LONGER VALID

1980's:

- Bernstein-Book (1983): $\alpha_{comp} > \alpha_{incomp.}$
- Baker (1983): compressibility either enhance or decrease α
- Sharp (1984): $\alpha_{comp} < \alpha_{incomp.}$

Since .... very few analytical work

DIMENSIONAL ANALYSIS

INCOMPRESS. $[g] = LT^{-2}$, $[\lambda] = L$ $([\rho] = ML^{-3})$

$\rightarrow$ no dimensionless number

$\rightarrow$ characteristic time $\begin{cases} \tau = \sqrt{\lambda / g} \\ \alpha^2 \sim kg \end{cases}$

$[\rho]$? M should disappear

$\rho_1 \rightarrow \rho_2$ $\quad R \equiv \rho_1 / \rho_2 \longrightarrow$ At.

COMPRESSIBLE : $[g] = LT^2$, $[\lambda] = L$, $[c_s] = LT^{-1}$

π - Theorem : $\pi = c_s^2 / (\lambda g)$

$$\begin{cases} c_s \text{ finite} \\ \lambda \rightarrow 0 \end{cases} \simeq \begin{cases} c_s \rightarrow \infty \\ \lambda \text{ finite} \end{cases} \quad (\pi \rightarrow \infty)$$

small λ's $(\lambda \ll c_s^2 / g)$ not affected by compress.

(Sharp 1983)

3 QUANT. REPRESENT. of. COMPRESS.

	$c_s^2 < \infty$	$\nabla \rho$	θ'
STATIC $(g = cste)$	×	×	
		×	
DYNAM.	×		×

neglected if $\lambda \ll (\nabla \rho / \rho)^{-1}$

scale of stratif.

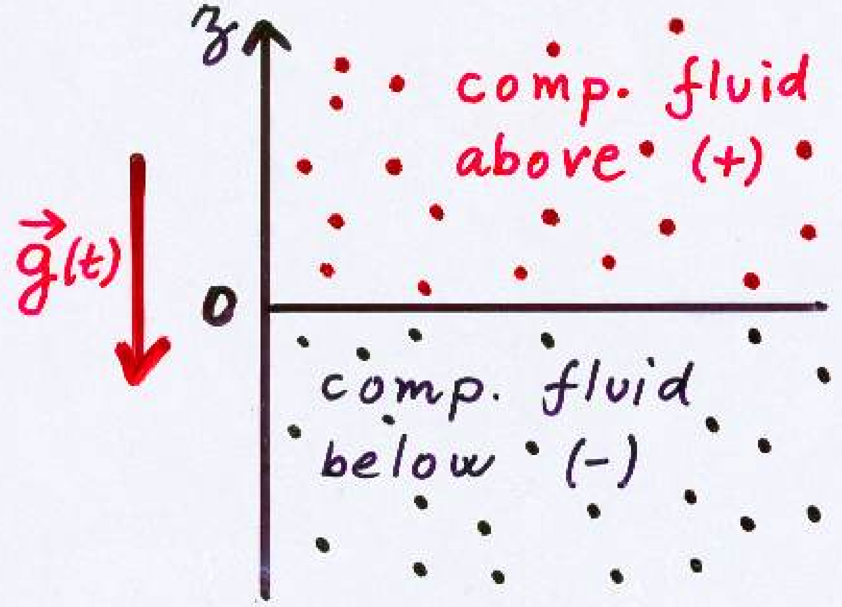
$\nabla \rho$: stratif

$\theta' \equiv \frac{\partial v}{\partial z}$

exp. velocity

$\vec{\nabla} . \vec{v} \neq 0$

ASSUMPTIONS



(x, y) 2D interface

$$\begin{cases} \dot{g} = 0 & \text{STATIC} \\ \dot{g} \neq 0 & \text{DYNAM.} \\ \rightarrow (\theta' \equiv \partial V / \partial z) & \end{cases}$$

<u>Assume:</u>
- initial state "0" $(\vec{V}_0 = 0$ or $\vec{V}_0 \neq 0)$
- no vorticity $\vec{V} = \vec{\nabla}\phi$

  $\frac{d}{dt}\int \vec{v} \cdot d\vec{\ell} = \text{cste} = \frac{d}{dt}\int (\vec{\nabla} \wedge \vec{V})\, d\vec{S} = \vec{0}$ if $\vec{V}_0 = 0$

  $\Rightarrow \vec{\nabla} \wedge \vec{V} = 0 \Rightarrow \vec{V} = \vec{\nabla}\phi$

  Idem $V_0 \neq 0$
- arbitrary EOS : $p^{\pm} = f^{\pm}(\rho^{\pm})$

  especially $p \sim \rho^{\gamma}$, $\gamma = \frac{1+n}{n}$ $(\gamma^{+} \neq \gamma^{-})$

<u>For "+" and "−", hydro. Eqs. are:</u>

$$\begin{cases} \rho \nabla^2 \phi + \vec{\nabla}\phi \cdot \vec{\nabla}\rho + \partial\rho/\partial t = 0 \\ \partial\phi/\partial t + (\vec{\nabla}\phi)^2/2 + g z + \underline{h} = 0 \end{cases}$$

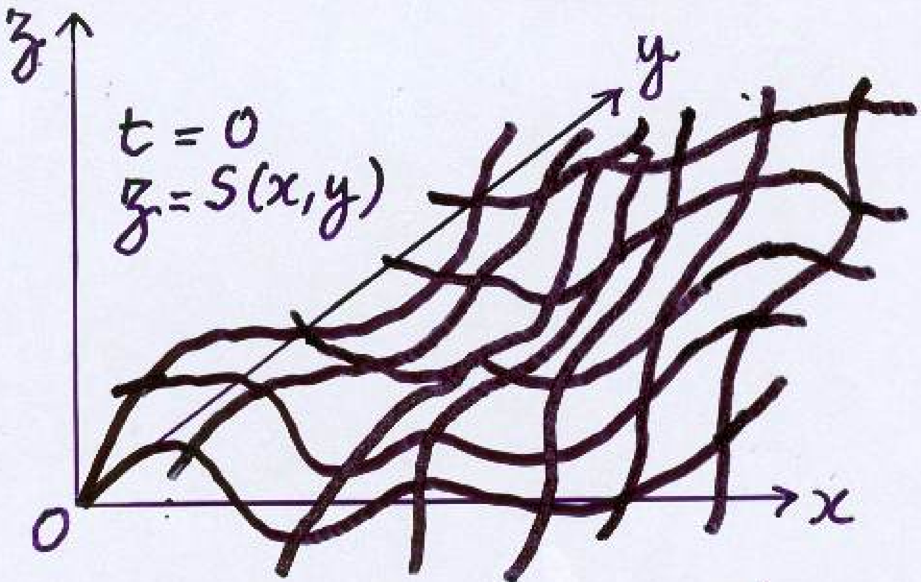
$$\begin{cases} \underline{h} : \text{enthalpy} \\ dh = dp/\rho = f'(\rho)\, d\rho/\rho \\ \text{Id. Gas}: h = c_s^2/(\gamma - 1) \end{cases}$$

THE PERTURBATION (STATIC)

Phys. quantity : $q(x,y,z;t) = \underbrace{q_0(z)}_{\text{Eq.}} + \underbrace{q_1(x,y,z;t)}_{\text{1rst o. pert.}}$

$\underbrace{\textbf{3D+t}}_{\text{4 dimensions!!!}}$ analytical stability analysis

Velocity potential: $\boxed{\phi(\vec{r};t) = \phi_{dim}\, S(x,y)\, Z(z)\, T(t)}$



$z = S(x,y)$

3D-interface

LINEAR SYSTEM OF 3-ODE'S:

$$\begin{cases} \bullet\ \ddot{T}(t) - \alpha^2 T(t) = 0 \Rightarrow T(t) = e^{\alpha t}; \quad \alpha = \text{G.Rate} \\ \qquad \text{Exponential (OK!)} \\ \bullet\ \partial_x^2 S(x,y) + \partial_y^2 (x,y) = -k^2 S(x,y) \\ \qquad k^2\text{: eigen values of } 2D\text{-}\nabla^2 \\ \bullet\ c_0^2(z)\, Z''(z) - \underset{\uparrow\, \dot{g}=0}{g}\, Z'(z) - \left[k^2 c_0^2(z) + \alpha^2\right] Z(z) = 0 \ ; \ c_0 = \frac{dp}{d\rho} = f'(\rho) \end{cases}$$

NO EXPONENTIAL SOLUTION !!!

$\longrightarrow$ YET DISPERSION RELATION

ISOTHERMAL CASE $p^{\pm} = K^{\pm}\rho^{\pm}$

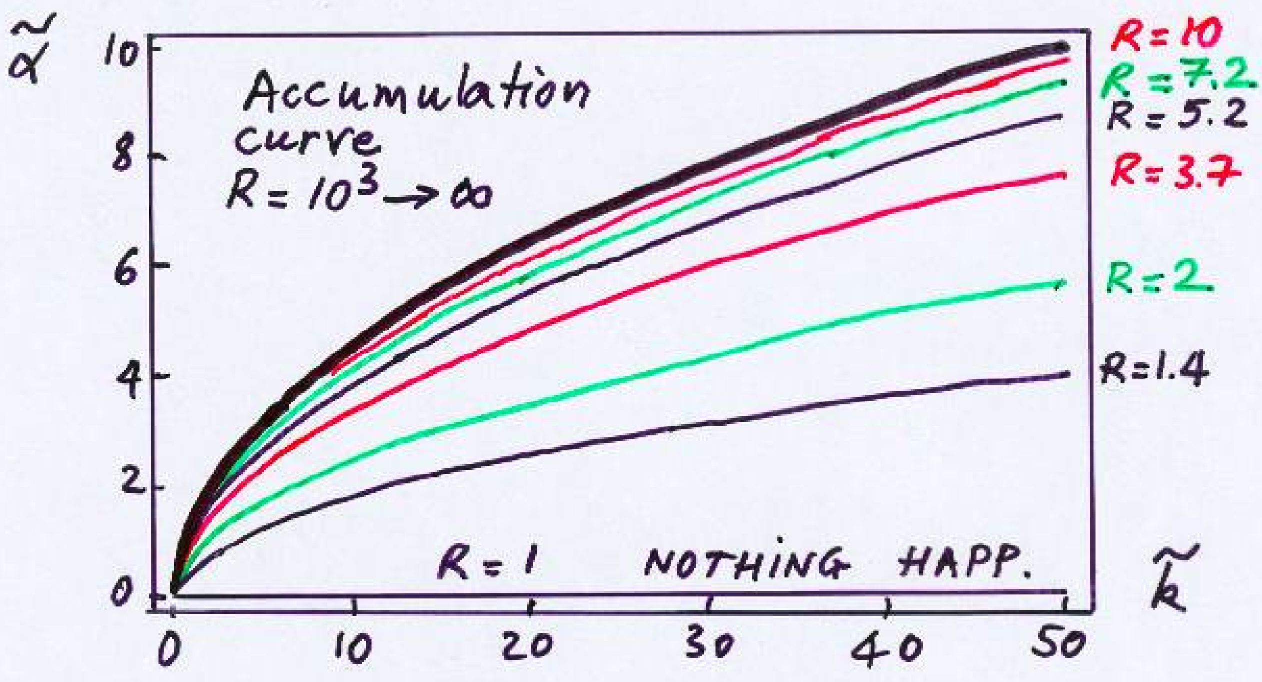
$$\tilde{\alpha}^2 = 2\left(\frac{1}{R} - 1\right) \frac{\left[1 - (1+\tilde{\alpha}^2+\tilde{k}^2)^{1/2}\right]\left[1 + (1+R\tilde{\alpha}^2+R^2\tilde{k}^2)^{1/2}\right]}{(1+\tilde{\alpha}^2+\tilde{k}^2)^{1/2} + (1+R\tilde{\alpha}^2+R^2\tilde{k}^2)^{1/2}}$$

IMPLICIT !

$$\begin{cases} \tilde{\alpha} = \alpha / (g/2\sqrt{K^+}) \\ \tilde{k} = k/(g/2K^+) \end{cases}$$ dimensionless

UNIVERSAL 1-PARAMATER R - CURVES

$R = \rho^+/\rho^-$ instead of $At = (R-1)/(R+1)$



$$\begin{cases} \bullet\ R \gg 1 \ (At \to 1): & \alpha = \sqrt{kg} \\ \bullet\ \tilde{k} \gg 1 \ (\lambda \to 0, c_{s0} \to \infty): & \alpha = \sqrt{At\, kg} \\ \bullet\ \tilde{k} \ll 1 \ (\lambda g / c_{s0}^2 \to \infty): & \alpha = \sqrt{2At}\, c_{0s}^{+} k \sim \frac{1}{\lambda} \end{cases}$$

incompress. $\alpha_{inc} \sim \frac{1}{\sqrt{\lambda}}$

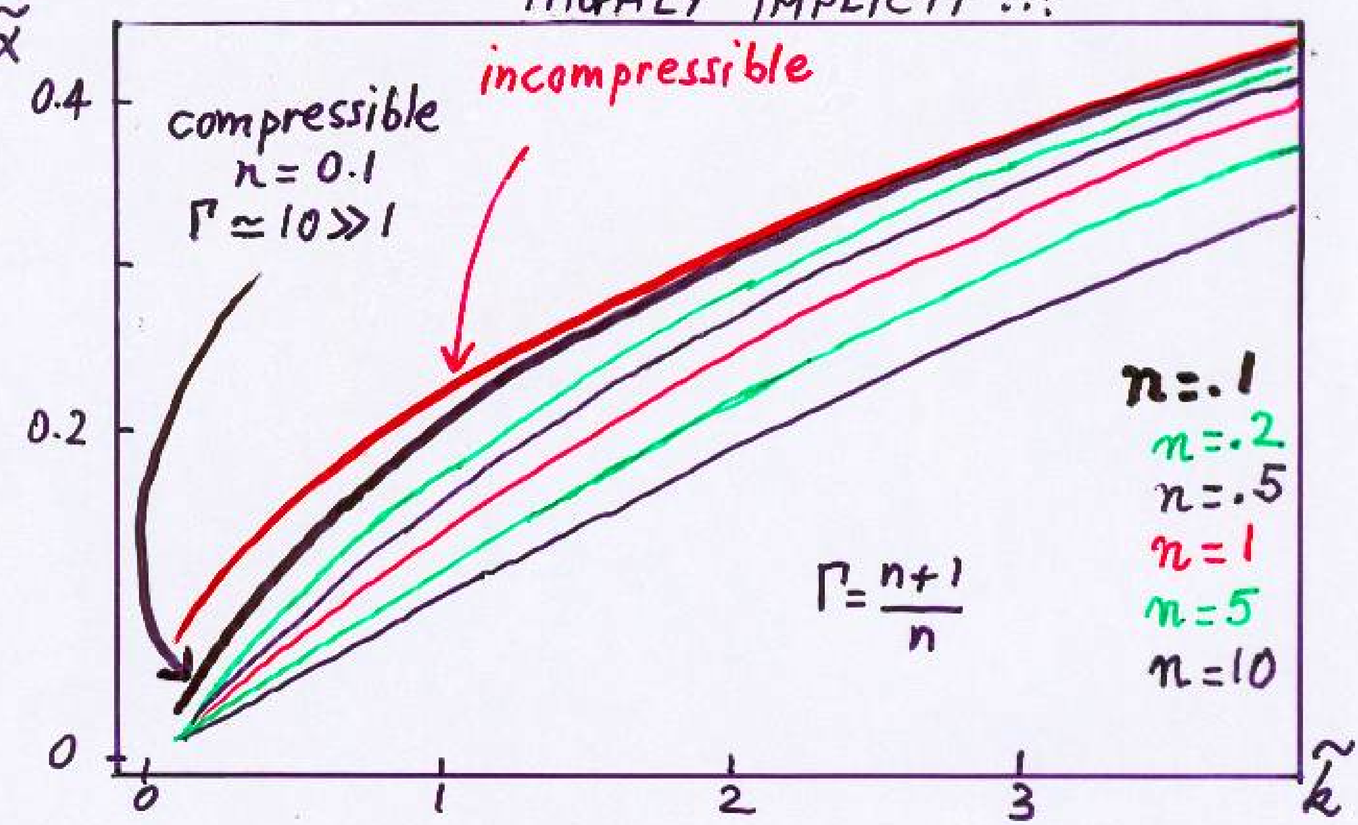
$\alpha_{comp} < \alpha_{incomp.}$

POLYTROPIC CASE $p^{\pm} = K^{\pm}(\rho^{\pm})^{\Gamma^{\pm}}$

2 parameters (R, Γ) universal family curves

$R = \rho^+/\rho^- = 1.1$ $(At \# 0.05)$

HIGHLY IMPLICIT !!!



- $\underline{\tilde{k} \gg 1}$: $\alpha \to \sqrt{At\, k g}$
- $\underline{n \to 0}$, $\underline{\Gamma \to \infty}$: — ≠ — $\tilde{k} < 2$

  large λ $\alpha_{comp.} < \alpha_{incomp.}$ (— below —)
- $\underline{\text{comparison}}$: $\tilde{k} = 2$

$$\begin{cases} \tilde{\alpha}_{comp} \simeq .15 \\ \tilde{\alpha}_{inc} \simeq .3 \end{cases}$$

$\sim 50\%$ deviation !!!
(and into an exponential)

DYNAMICAL CASE
POLYTROPE Γ
$\dot{g}/g \ll \alpha$

1-D background motion $\vec{v} = v(z,t)\,\vec{e}_z$ and $\theta' = \partial v/\partial z$
$(\vec{\nabla}.\vec{v} \neq 0)$

$$\alpha(\alpha - \theta')$$
$$= g^2(\rho^+ - \rho^-)\frac{\{[1+4c_0^+(\alpha^2+\alpha a^+ + c_0^{+2}k^2)/g^2]^{1/2} - 1\}\{-\text{superscript}\}}{2\rho^+ c_0^{+2}\{-\} + 2\rho^- c_0^{-2}\{+\}}$$

$$a^\pm = \theta'(\Gamma - 1)$$

ASYMPTOTICS :

- $k \to \infty$: $\alpha(\alpha - \theta') \to \alpha^2_{incomp} = A_t k g$

$$\alpha \to \frac{\theta' + \sqrt{\theta'^2 + 4\alpha_{inc}^2}}{2} \qquad \theta' = 0,\ \alpha = \alpha_{inc}$$

$\begin{cases} \theta' > 0 \text{ (expansion)} & \alpha > \alpha_{inc} \\ \theta' < 0 \text{ (contraction)} & \alpha < \alpha_{inc} \end{cases}$

- $c_0^\pm \to \infty$: same as for $k \to +\infty$.

CONCLUSION

- Compressibility requires careful study (comparison with incomp. should continued)
- Equilibrium initial State :

$\alpha_{comp} < \alpha_{incomp} = \sqrt{A_t k g}$

Similar to $\alpha_\nabla \simeq \sqrt{\dfrac{A_t k g}{1 + k L_\nabla}}$

Stratification due to compressibility similar to the existence of a $\vec{\nabla}\rho$

- Incompressible:

$\vec{V} = \vec{\nabla}\Phi$ and $\vec{\nabla}.\vec{V} = 0 \Rightarrow \nabla^2 \Phi = 0$

<u>Compressible</u> $\phi = \phi_{dim}\, S(x,y)\, Z(z)\, T(t)$

with $\nabla^2 S = -k^2 S$

Similar to $\nabla^2 \Phi = 0$, but ...

... additional equation for $Z(z)$, $T(t)$!

- To be used as a starting point for :
  - Non Linear Study
  - Comparison with codes.