Response of Turbulent RANS Models to Self-Similar Variable Acceleration RT-Mixing: an Analytical "0D" Analysis

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Abstract

So far, the validation of RANS models applicable to variable acceleration RT mixing flows (as found in ICF) has mostly been carried out by fitting experimental or numerical data obtained for constant (RT) and impulsive (RM) accelerations. Further checks are also possible on the few available data for variable acceleration, such as in mixing-demixing flows induced by reverting the gravity field. Although this approach is widely applied and accepted, it is unsatisfactory because of the complex relationship between the model features and coefficients and the experimental measurements.

It is shown here that self-similar variable acceleration RT (SSVART) provides an appealing alternative since it extends the usual calibration techniques of turbulent RANS models based on simple self-similar flows. The general model equations in 1D (PDEs) are still too complex for full analytical calculations of SSVART flows, but using reasonable assumptions, simple 0D (ODEs) approximations can be derived and solved analytically.

This approach is applied to an extended k–ε model derived from Andronov's (1979) and to Youngs' (1989) two-fluid model. The behaviour of the mixing layer growth rate and integral turbulent scales provides important informations on the accuracy of these models.

Finally, general qualitative arguments will be discussed showing the importance and the difficulty of capturing accurately a broad range of SSVARTs with a single simple model.

I Introduction

Qualitative understanding of large scale turbulence
in Rayleigh-Taylor (RT)
and incompressible Richtmyer-Meshkov (IRM)
is not available as in other mixing layers (wakes, jets...).

Our aim is to provide such analysis for:

- ideal incompressible self-similar turbulent regime,
- vanishingly small density differences ($\mathcal{A} \to 0$),
- and "0D" (global) energy balance.

"0D" approximation is averaging over TMZ width, $L(t)$.

Two-fluid analysis will be applied.

Kelvin-Helmoltz (KH) will be used as reference.

Experimental, simulated and modeled results will be examined.

Extension to SSVARTs will be considered
(Self-Similar Variable Acceleration Raleigh-Taylor mixing).

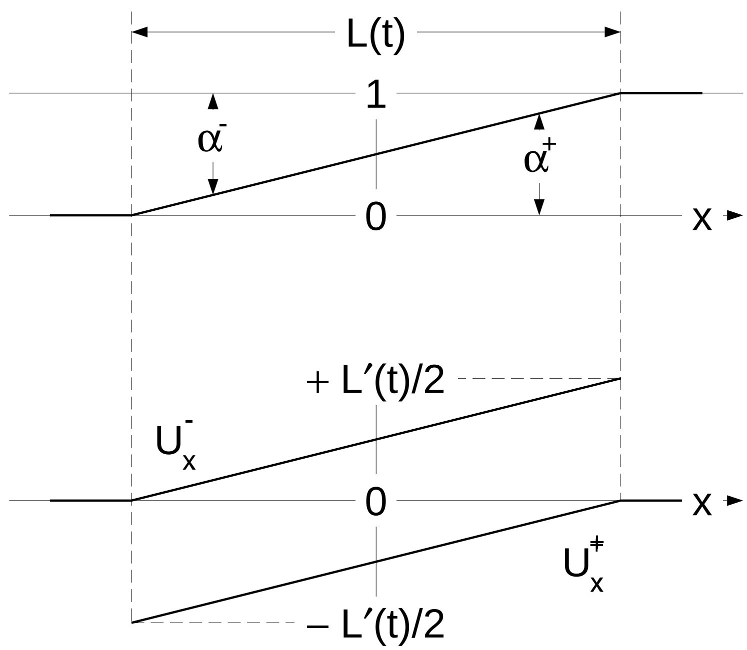
II "0D" averaging

II.1 Fluid motion

In $\mathcal{A} \to 0$ limit

experiments and simulations show

linear profiles of fluid volume fractions:



From mass conservation equations for fluids $+$ and $-$,

fluid velocities $U^{\pm}$ are also found linear.

Therefore:

$$\delta U_x = U_x^+ - U_x^- = \frac{\mathrm{d}}{\mathrm{d}t} L(t) \tag{1}$$

II.2 TMZ-growth related energies

From growth law and fluid velocities, one can extract:

- Total input of energy:

$$K_I = \frac{-\,2\,g}{(\rho^+ + \rho^-)\,L} \left(\int_{-L/2}^{+L/2} (\alpha^+(x)\rho^+ + \alpha^-(x)\rho^-)\,x\,\mathrm{d}x - \int_{-L/2}^{0} \rho^- x\,\mathrm{d}x - \int_{0}^{+L/2} \rho^+ x\,\mathrm{d}x \right). \qquad (2)$$

- Mean kinetic energy:

$$\begin{aligned} K_M &= \int_{-L/2}^{+L/2} \overline{\rho}(U_x)^2\!/2\,\mathrm{d}x \Big/ \int_{-L/2}^{+L/2} \overline{\rho}\,\mathrm{d}x \\ &\approx \tfrac{1}{15}\,\mathcal{A}^2(\delta U_x)^2 \quad \text{for } \mathcal{A} \to 0. \end{aligned} \qquad (3)$$

- Directed kinetic energy:

$$\begin{aligned} K_D &= \int_{-L/2}^{+L/2} [\alpha^+\rho^+(U_x^+)^2 + \alpha^-\rho^-(U_x^-)^2 - \overline{\rho}(U_x)^2]/2\,\mathrm{d}x \Big/ \int_{-L/2}^{+L/2} \overline{\rho}\,\mathrm{d}x \\ &\approx \tfrac{1}{12}\,(\delta U_x)^2 = \tfrac{1}{48}\,(L')^2 \quad \text{for } \mathcal{A} \to 0. \end{aligned} \qquad (4)$$

K_M will thus be ignored.

II.3 Basic self-similar growth laws

Instability	RT	IRM	KH
$L(t)$	$\mathcal{Y}_0 \times \mathcal{A}gt^2$	$L_0 \left(\frac{t}{t_0}\right)^{n_0}$	$\mathcal{X}_0 \times \Delta U_y t$
K_I	$\frac{\mathcal{Y}_0}{12} \times (\mathcal{A}gt)^2$	$K_{I0} \left(\frac{t}{t_0}\right)^{-n_0}$	$\frac{1}{12} \times (\Delta U_y)^2$
K_D	$\frac{\mathcal{Y}_0^2}{12} \times (\mathcal{A}gt)^2$	$\frac{n_0^2}{48} \times \frac{L_0^2}{t_0^2}\left(\frac{t}{t_0}\right)^{-2(1-n_0)}$	$\frac{\mathcal{X}_0^2}{48} \times (\Delta U_y)^2$
K_D/K_I	$\mathcal{Y}_0$	$\frac{n_0^2}{48} \times \frac{L_0^2}{K_{I0}\, t_0^2} \left(\frac{t}{t_0}\right)^{-(2-3n_0)}$	$\frac{\mathcal{X}_0^2}{4}$

Since $\mathcal{Y}_0 \approx \mathcal{X}_0 \approx 0.1$,

K_D/K_I levels largely motivate use of two-fluid models for RT.

II.4 Energy balance

The knowledge of the total kinetic energy in TMZ, K,
obtained from experiments or simulations,
completes balance and provides dissipation, E.

The ratio of integral length scale to TMZ width:

$$\kappa_T = \frac{K_T^{3/2}}{E\,L}, \tag{5}$$

will be called **Von Kármán number**

by analogy with the Von Kármán Constant of turbulent boundary layers,

with the two-fluid turbulence defined as:

$$K_T = K - K_D \tag{6}$$

Instability	RT	IRM	KH
Growth coefficient	$\mathcal{Y}_0$=0.12	n_0=0.3	$\mathcal{X}_0$=0.1
K_T/K_D	3	58	116
κ_T	0.09	0.3	0.63

see references in Llor 2001.

The Von Kármán number is 7 times smaller in RT than in KH:
assuming there is one large eddy in KH TMZ,
there are seven large eddies in RT TMZ !

(confirmed in experiments or simulations, Dimonte 2000, Dalziel 1999, Inogamov 2001...)

III Results from "0D" averaged models

III.1 "0D" averaging of model PDEs

From equation for hydro quantity a in model:

$$\frac{\partial}{\partial t}a + (\phi_x^a)_{,x} = s^a, \tag{7}$$

averaging over TMZ width, $A = \langle a \rangle$, yields:

$$\frac{1}{L}\frac{\mathrm{d}(LA)}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\mathrm{d}L}{L\,\mathrm{d}t}A = S^a, \tag{8}$$

since (usually) $\phi_x^a = 0$ at edges of TMZ.

Supplementary asumptions/approximations are needed to provide closed expression of S^a.

For instance, center of TMZ is often dominant, so:

$$S^a \approx \frac{1}{\zeta}\, s^a\left(\zeta\, B\right), \tag{9}$$

if $s^a = s^a(b)$ and defining $\zeta \approx a(0)/A \approx b(0)/B \approx 3/2$.

Models are thus reduced from PDEs to ODEs.

Two models considered here:

- Andronov's modified single-fluid k–ε model (1979),
- Youngs' two-fluid model (1989).

III.2 PDEs of modified k–ε and two-fluid models

Basic relevant PDEs for Andronov's modified k–ε:

$$\begin{cases} \frac{\partial}{\partial t}\,\overline{\rho c^{\pm}}\ \dots\ =\ \dots - (\overline{\rho c^{\pm} u_i^{"}})_{,i} \overset{\mathsf{m}}{=} \left(\overline{\rho}\frac{\nu_t}{\sigma_c}\widetilde{c^{\pm}}_{,i}\right)_{,i} & \Rightarrow \frac{\mathsf{d}}{\mathsf{d}t}L(t) \\ \frac{\partial}{\partial t}\,\overline{\rho k}\ \dots\ =\ \dots - \overline{p}_{,i}\overline{u_i^{"}} \overset{\mathsf{m}}{=} -\frac{\nu_t}{\sigma_\rho}\frac{\overline{p}_{,i}\overline{\rho}_{,i}}{\overline{\rho}} & \Rightarrow \text{“RT” prod.} \\ \frac{\partial}{\partial t}\,\overline{\rho\epsilon}\ \dots\ =\ \dots \qquad \overset{\mathsf{m}}{=} -C_{\epsilon 0}\frac{\nu_t}{\sigma_\rho}\frac{\overline{p}_{,i}\overline{\rho}_{,i}}{\overline{\rho}} & \end{cases}$$

with $\sigma_\rho \approx 2$ and $C_{\epsilon 0} \approx 0.85$ (other constants as usual).

Basic relevant PDEs for Youngs' two-fluid:

$$\begin{cases} \frac{\partial}{\partial t}\,(\alpha^{\pm}\rho^{\pm})\ \dots\ =\ \dots & \\ \frac{\partial}{\partial t}\,(\alpha^{\pm}\rho^{\pm}U_i^{\pm})\ \dots\ =\ \dots - (R_{ij})_{,j} \mp D_i(\lambda_d) & \Rightarrow \frac{\mathsf{d}}{\mathsf{d}t}L(t) \\ \frac{\partial}{\partial t}\,(\overline{\rho}k_b)\ \dots\ =\ \dots\ \pi_{\mathsf{KH}} + D_i\delta U_i & \\ \frac{\partial}{\partial t}\,\lambda_d\ \dots\ =\ \dots\ \sqrt{\frac{2\overline{\rho}}{\rho^{+}+\rho^{-}}}n_i\delta U_i & \Rightarrow \text{Length scale} \end{cases}$$

$$D_i \overset{\mathsf{m}}{=} C_d\overline{\rho}\frac{\alpha^{+}\alpha^{-}}{\lambda_d}\|\, U_i^{+}-U_i^{-}-\vec{W}\,\|\,(U_i^{+}-U_i^{-}-W_i) + a.m. + \dots$$

$$\nu_t = C_\mu\sqrt{k_b}\,\lambda_i, \qquad \lambda_i = \frac{C_i}{C_\mu}\lambda_d,$$

with $C_d \approx 20$ and $C_i \approx 0.105$ (other constants as usual).

III.3 ODEs of modified k–ε and two-fluid models

"0D" ODEs for Andronov's modified k–ε:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}L = \frac{8\,\zeta\,C_\mu}{\sigma_c}\frac{K^2}{E\,L}, \\ \frac{\mathrm{d}}{\mathrm{d}t}K = -\frac{\mathrm{d}L}{L\,\mathrm{d}t}K + \Pi_K - E, \\ \frac{\mathrm{d}}{\mathrm{d}t}E = -\frac{\mathrm{d}L}{L\,\mathrm{d}t}E + C_{\epsilon\pi}\frac{E}{K}\Pi_K - C_{\epsilon 2}\frac{E^2}{K}, \end{cases} \tag{10}$$

$$\Pi_K^{\boxed{\mathrm{RT}}} \approx \frac{2C_\mu}{\sigma_\rho}\frac{K^2}{E\,L}\mathcal{A}\,\Gamma, \qquad \Pi_K^{\boxed{\mathrm{KH}}} \approx C_\mu\frac{K^2}{E}\left(\frac{\Delta U_y}{L}\right)^2.$$

"0D" ODEs for Youngs' two-fluid:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}L = 4\sqrt{3\,K_D}, \\ \frac{\mathrm{d}}{\mathrm{d}t}K_D = -\frac{\mathrm{d}L}{L\,\mathrm{d}t}K_D - \Pi_D + \frac{2}{3}\mathcal{A}\Gamma\sqrt{3\,K_D}, \\ \frac{\mathrm{d}}{\mathrm{d}t}K_T = -\frac{\mathrm{d}L}{L\,\mathrm{d}t}K_T + \Pi_D - \frac{2\,C_\mu}{C_i}\sqrt{\zeta}\frac{K_T^{3/2}}{L}, \end{cases} \tag{11}$$

$$\Pi_D = \left[\tfrac{C_d}{2\,L}\left(2\sqrt{3\,K_D} - 4C_i\sqrt{\zeta K_T}\right)^2 + \tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{\sqrt{3\,K_D}}{4} + \tfrac{3\,K_D}{2\,L}\right]\ \tfrac{2\sqrt{3\,K_D}}{\zeta}.$$

KH more complex but can be treated in similar way.

III.4 Self-similar solutions of model ODEs

Self-similar solutions are analytical for both models (explicit for k–ε),
and yield the following numerical results
(for recommended values of the coefficients):

Source	Exp. & Sim.			Modified k–ε			Young's two-fluid		
Instability	RT	IRM	KH	RT	IRM	KH	RT	IRM	KH
$\mathcal{Y}_0$, n_0, $\mathcal{X}_0$	0.12	0.3	0.1	0.054	0.3	0.077	0.106	0.44	0.105
K_T/K_D	3	58	116	4.9	117	106	6.2	26	93
κ_T	0.09	0.3	0.63	1.4	0.42	0.53	0.58	0.58	0.76

Deviations below 30% due to exp. errors, "0D" averaging and modeling.
However, significant discrepancies in RT:
- $\mathcal{Y}_0$ is 50% too low in k–ε,
- κ_T is too high by factors 15 and 6 in k–ε and two-fluid respectively.

III.5 Conclusion on models

Inconsistencies for k–ε due to purely diffusive TMZ growth.

In k–ε, directed transport in RT
is replaced by enhanced diffusion (sufficient for KH and IRM).

$L(t)$ equation in (10) for RT case can be rewritten as:

$$1 = \frac{2\,\zeta\,C_\mu}{\sigma_c}\,\kappa\,\sqrt{\frac{K}{3K_D}}, \tag{12}$$

which yields unphysically small value $\sigma_c \approx 10^{-2}$,
for known estimates of κ and $\sqrt{K/K_D}$.

In two-fluid model,
production term of λ is $\propto \delta U$ ($\approx$ uniform over TMZ).
Therefore, $\kappa_T = \langle\lambda\rangle/L = C_i/(2C_\mu) \approx 0.58$ is constant.

Variations in κ cannot be reproduced;
but it has little consequence since
diffusion is properly captured in IRM and KH,
whereas transport is dominated by directed energy in RT.

IV Expanding horizons: SSVARTs

IV.1 Definition and usefulness of SSVARTs

Self-Similar Variable Acceleration Rayleigh-Taylor flows,
or SSVARTs, defined by acceleration profiles:

$$\begin{array}{llll} \text{if} & n > -2, & g \propto t^n & \text{for} \;\; t > 0, \\ \text{if} & n < -2, & g \propto (-t)^n & \text{for} \;\; t < 0. \end{array} \tag{13}$$

Since $L(t) \propto t^{n+2}$,
growth of $L(t)$ demands the $-t$ inversion for $n < -2$.

May be out of reach from experiments,
but can be studied numerically (Youngs, this conference).

Will be assumed to exist as Reynolds (ensemble) averages.

Response will be:

$$L(t) = \mathcal{Y}_n \mathcal{A}\, g(t)\, t^2 \quad \text{where} \quad \mathcal{Y}_n > 0. \tag{14}$$

Testing models in variable acceleration is difficult,
but SSVARTs are helpful in testing robustness and accuracy,
since they represent a "basis" set of all possible $g(t)$.

IV.2 Stability of SSVARTs

Self-similar $g(t)$ can always be considered,
but response of system is VART that may not be SSVART.

For stability (and existence),
a SSVART must be an attractor of motion
and eventually forget its initial condition.

Behavior of input energy provides criterion:

- $g(t)$ energy input from t_0 to t behaves as $K_I \propto (\pm t)^{2n+2}$,
- initially present energy at t_0 is dilluted as $K_0 \propto (\pm t)^{-(n+2)}$.

Therefore, initial condition is not forgotten if:

$K_0/K_I \to \infty$ for $\pm\, t \to \infty$ or 0,

which yields $-2 < n < -4/3$.

Since K_0 is both diluted and dissipated during growth,
the forbiden interval is less restrictive :

$$-2 < n < -2 + n_0, \tag{15}$$

Within this range of "forbidden" values,
SSVARTs eventually behave as Richtmyer-Meshkov, $L(t) \propto t^{n_0}$
(the evolution is dominated by the initial energy).

IV.3 SSVARTs as seen by "0D" models

- For Andronov's modified k–ε:

$$\mathcal{Y}_n \;=\; \frac{2C_\mu}{\sigma_\rho}\frac{(C_{\epsilon 2}-C_{\epsilon 0})^2}{[(3C_{\epsilon 2}-3)n+(4C_{\epsilon 2}-3)][(3C_{\epsilon 0}-3)n+(4C_{\epsilon 0}-3)]}. \quad (16)$$

- For Youngs' two-fluid model:

$$\mathcal{Y}_n \;=\; \frac{1/(1+n/2)}{C_d\left(1-\sqrt{2}C_i\,\mathcal{U}_n\right)^2(1+n/2)+3(1+3n/4)/2}, \quad (17)$$

where $\mathcal{U}_n=\sqrt{K_T/K_D}$ is largest real solution of:

$$C_\mu\,\mathcal{U}_n^3 \;-\; 4\sqrt{2}C_i\left[2C_i^2C_d-\frac{4+3n}{2(2+n)}\right]\mathcal{U}_n^2 \;+\; 16C_i^2C_d\,\mathcal{U}_n \;-\; 2\sqrt{2}C_i\left[C_d+\frac{4+3n}{2(2+n)}\right] \;=\; 0. \quad (18)$$

- For scaled free fall

  (obtained by canceling Π in two-fluid K_D equation):

$$\mathcal{Y}_n \;=\; \frac{\mathcal{Y}_0}{(1+n/2)(1+3n/4)}, \quad (19)$$

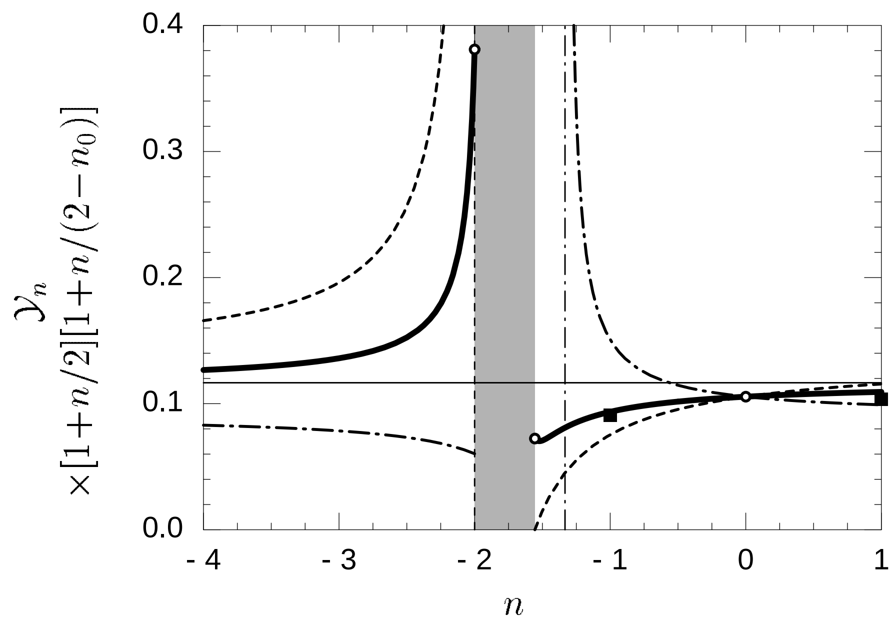
- For Read's formula ($L\propto\mathcal{A}\left(\int\sqrt{g}\right)^2$):

$$\mathcal{Y}_n \;=\; \frac{\mathcal{Y}_0}{(1+n/2)^2}, \quad (20)$$

All models generate two main poles for $\mathcal{Y}_n$.

k–ε provides negative values outside pole interval $[-1.7, 0.89]$
for recommended values of σ_ρ and $C_{\epsilon 0}$.
Positive values outside interval $[-2, -1.7]$ requires $C_{\epsilon 0} \geq 3/2$,
but then $\mathcal{Y}_0 \approx 10^{-3}$!
Again k–ε cannot capture directed transport
which is enhanced as n is increased.

Behavior of all other models is similar (common $n = -2$ pole)
except near poles associated with IRM growth coefficients:



Normalized growth rates of various models:
Youngs' two-fluid (dot-dashed), free fall (dashed) Read's formula (dotted),
and Youngs' preliminary simulations presented at this conference (filled squares).

Free fall and Read's formula respectively provide
highest ($n = -4/3$) and lowest ($n = -2$) positions
of "IRM" pole.

Youngs' two-fluid model is intermediate
between free fall and Read's formula.

Free fall and Read's formula are good approximations around RT,
but can diverge severly in other regimes.

Three continuously connected regimes are found:

- $n_0 - 2 < n < -1$:
  directed transport is dominated by diffusion,
- $-1 < n$ and $n < -5/2$:
  directed transport is limited by turbulent viscosity,
- $-5/2 < n < -2$:
  directed transport is limited by free fall.

Single-fluid models seem limited to first regime,
approximate models are accurate to 30% over second regime,
but only two-fluid approaches can capture all three.

V Conclusions

RANS models can efficiently be analysed as "0D" averages,
without heavy numerical developments.

Such analysis show importance,
of energy balance and Von Kármán number
in characterizing transport and turbulence structure.

Two-fluid approaches seem superior to single-fluid approaches
in capturing directed transport with robust behaviour.

SSVARTs have been shown to provide
a useful set of benchmark flows for models.

Appropriate capture of SSVARTs seems to require two-fluid model.

RT and IRM are limiting cases of SSVARTs.

SSVARTs are currently being investigated by DNS/LES.

Extention of SSVARTs to complex n ?

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