Toy models for Rayleigh-Taylor instability:

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(9:10 – 9:30)

with thanks to

Joanne Holford (DAMTP)

David Youngs (AWE)

The growth question:

$$h = \alpha A g t^2, \text{ where } A = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

But what is α?

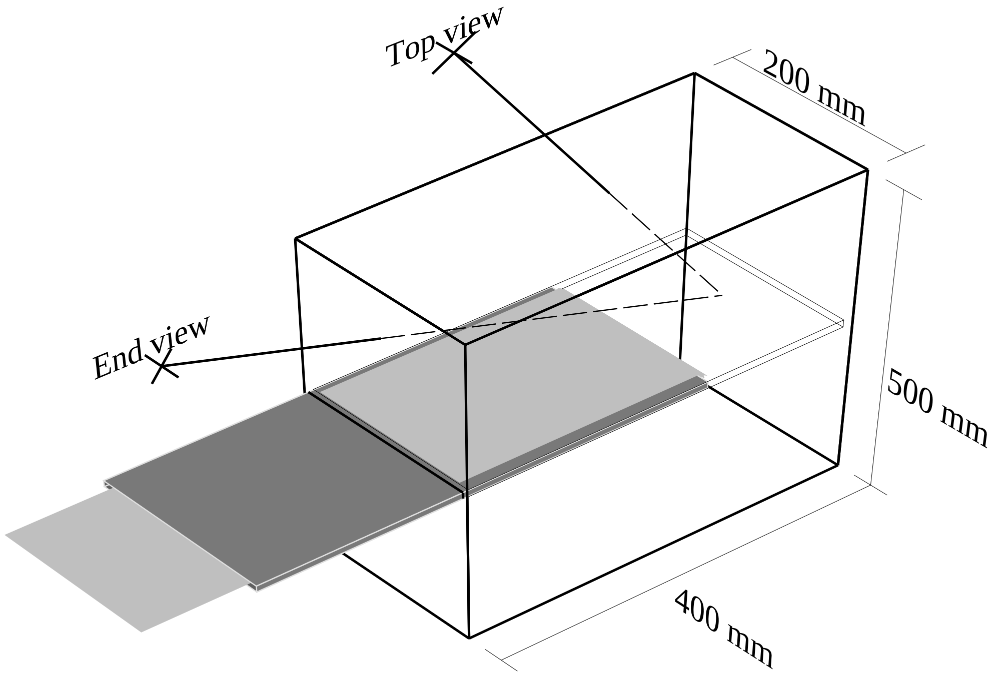
◊ 0.10, … 0.07, 0.06, … 0.03, 0.02 ?

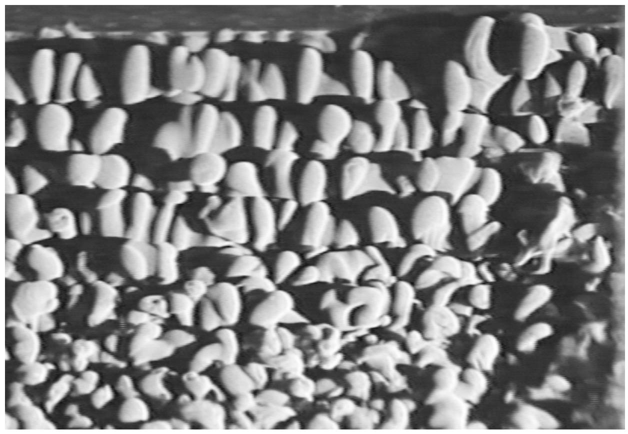
Timescale: $$T = \sqrt{\frac{H}{Ag}}$$

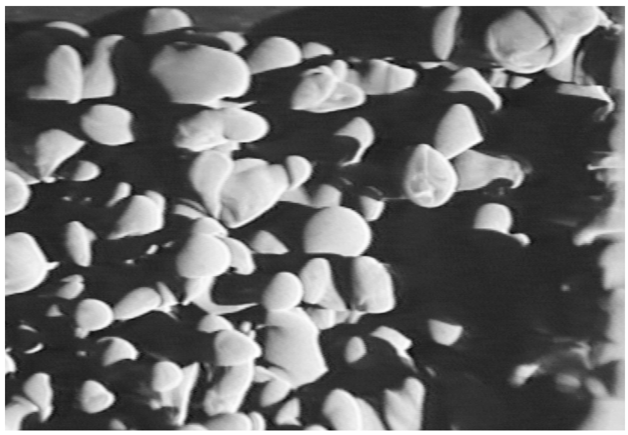
If $\delta = h/H$, and $\tau = t/T$,

then $$\delta = \alpha\,\tau^2$$

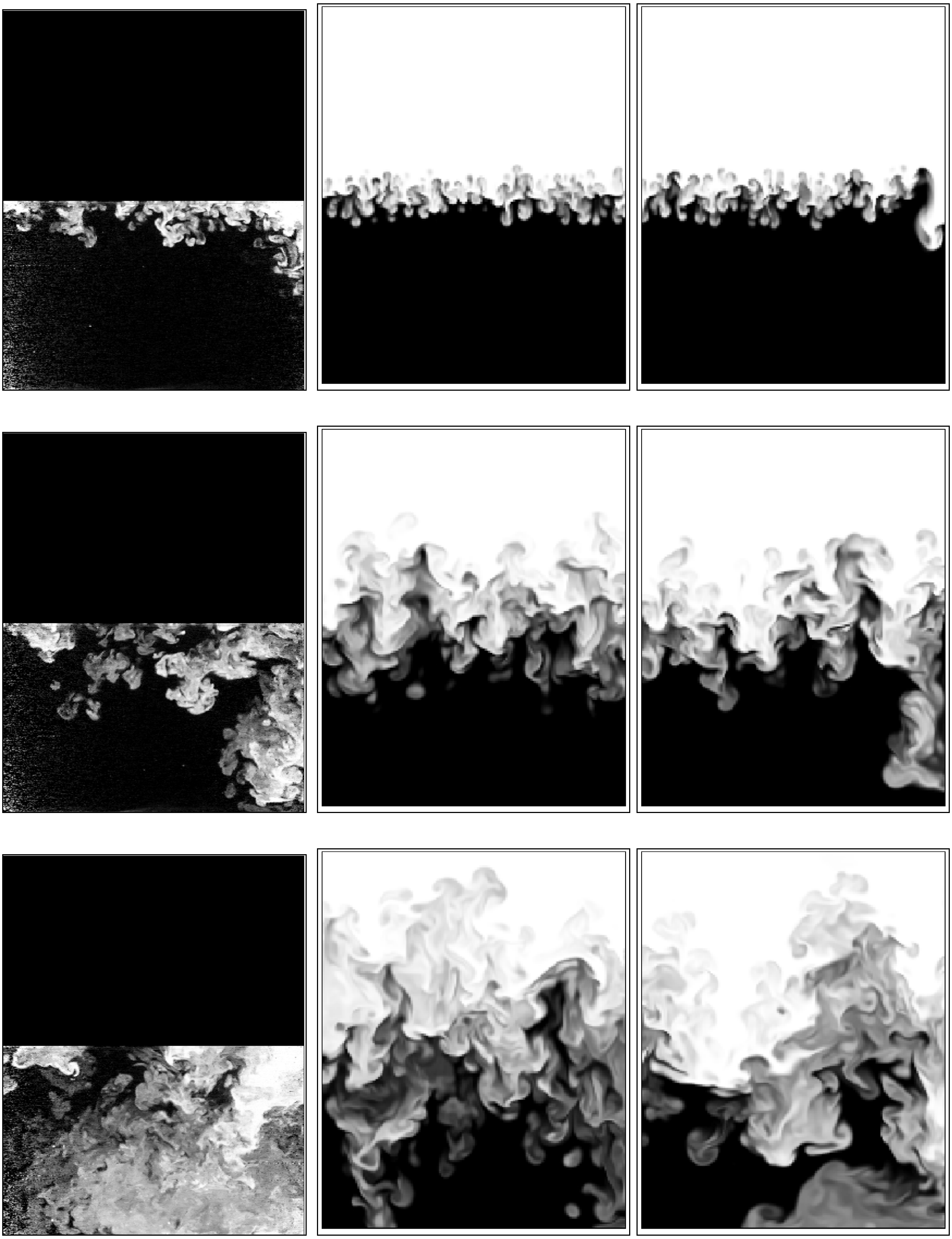
Experiments







Appropriate modelling (?)



Growth

Dimensional analysis/similarity theory

$$h = \alpha A g t^2.$$

Single mode

Layzer (1955)

For
$$\zeta(x,y) = a_0 \cos\frac{2\pi x}{\lambda}$$

if
$$\frac{dh}{dt} = w,$$

then
$$(2+E)\frac{dw}{dt} = Ag(1-E) - C_D\frac{w^2}{\lambda},$$

where
$$E = \exp\left(\frac{-6\pi h}{\lambda}\right).$$

Experimentally $C_D \sim 10$

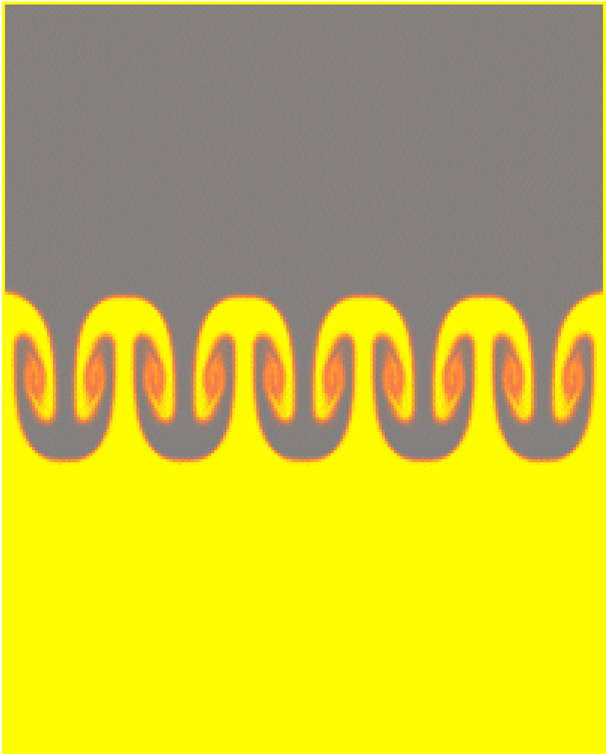
◊ Does this make sense?

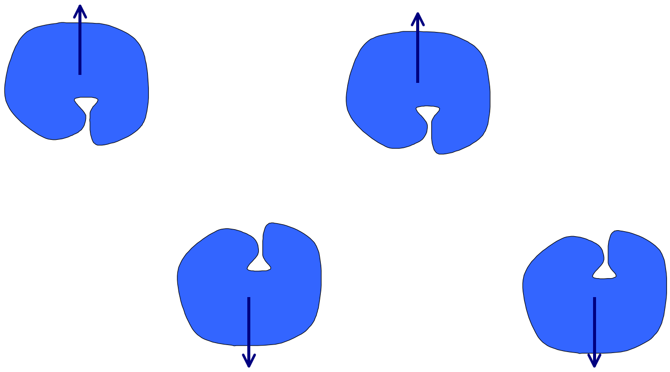
Early time → linear theory $\dfrac{d^2h}{dt^2} = \dfrac{2\pi A g}{\lambda} h$

Late time → constant velocity $w_\infty = \sqrt{\dfrac{Ag\lambda}{C_D}}$

$$\Rightarrow h \to w_\infty(t - t_0)$$

Structure

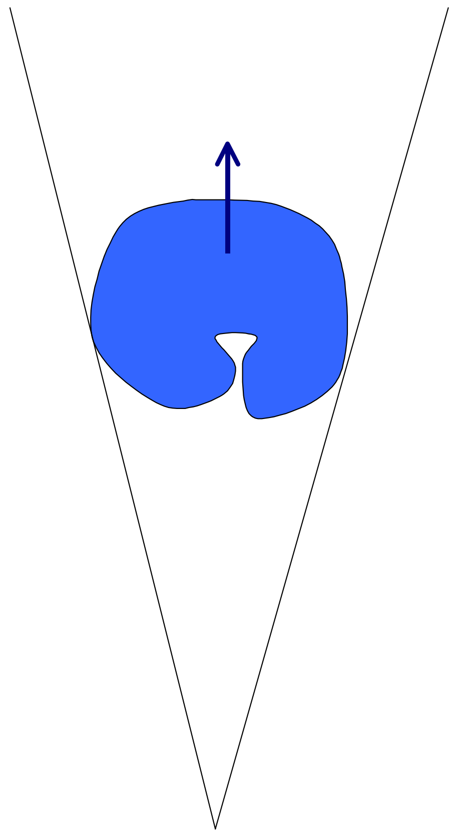




Often described as ‘bubbles’…

…but more like ‘thermals’ in miscible fluids

Thermals



Self-similar

$$r = \theta z$$
$$V = \gamma r^3.$$

Buoyancy conserved

$$g' V = g' \gamma r^3 = g'_0 V_0.$$

Constant Froude number

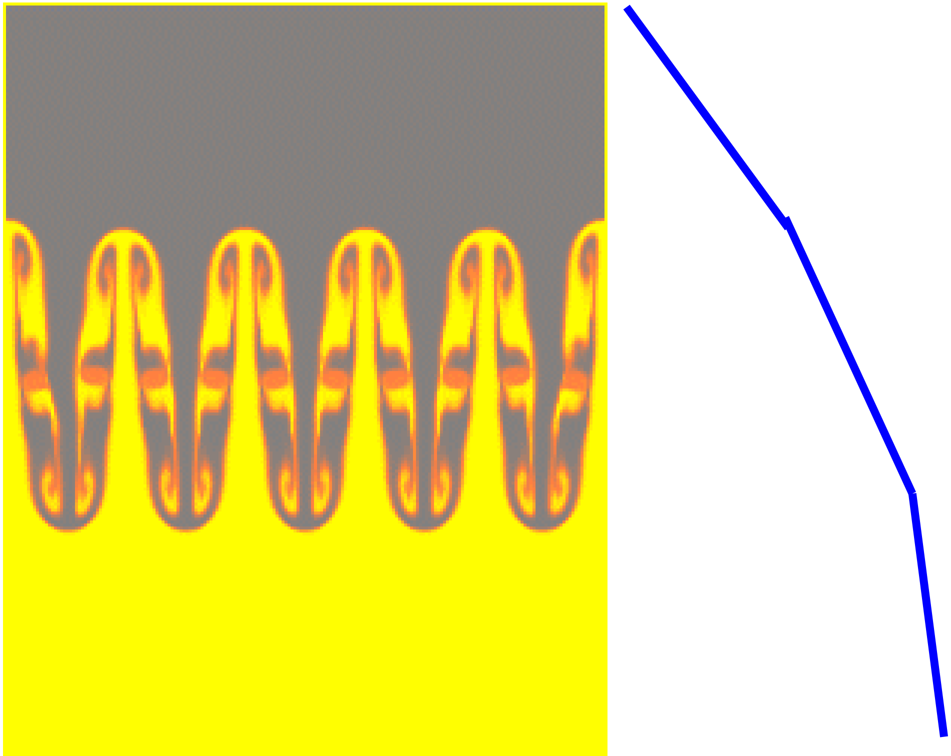
$$F^2 = \frac{w^2}{g' r}$$

Integrating $w = dz/dt$

$$\frac{\gamma^{1/2}\theta}{2F\left(g'_0 V_0\right)^{1/2}} z^2 = t$$

Experimental results $\rightarrow F \approx 1.2$.

Rayleigh-Taylor as thermals



Froude number ~ 1.2 (aspect ratio 0.72) $\Rightarrow C_{Thermal} \approx 1.3.$

Rayleigh-Taylor bubbles a little like thermals $\rightarrow C_D \approx 1.3$

But in Rayleigh-Taylor environment

- Density field not hydrostatic in ambient
  - ◊ Hydrostatic in mean density $\Rightarrow$ halve buoyancy force $\rightarrow C_D \approx 2.6$
- Flow around bubble affected by bubble moving in opposite direction
  - ◊ Drag due to twice rise speed of bubble $\rightarrow C_D \approx 10.4$

In agreement with single mode experiments

BUT natural R-T has more than one mode

Multi-mode

What happens if λ grows with h?

Let $\lambda = \psi h$

Late times approximation:

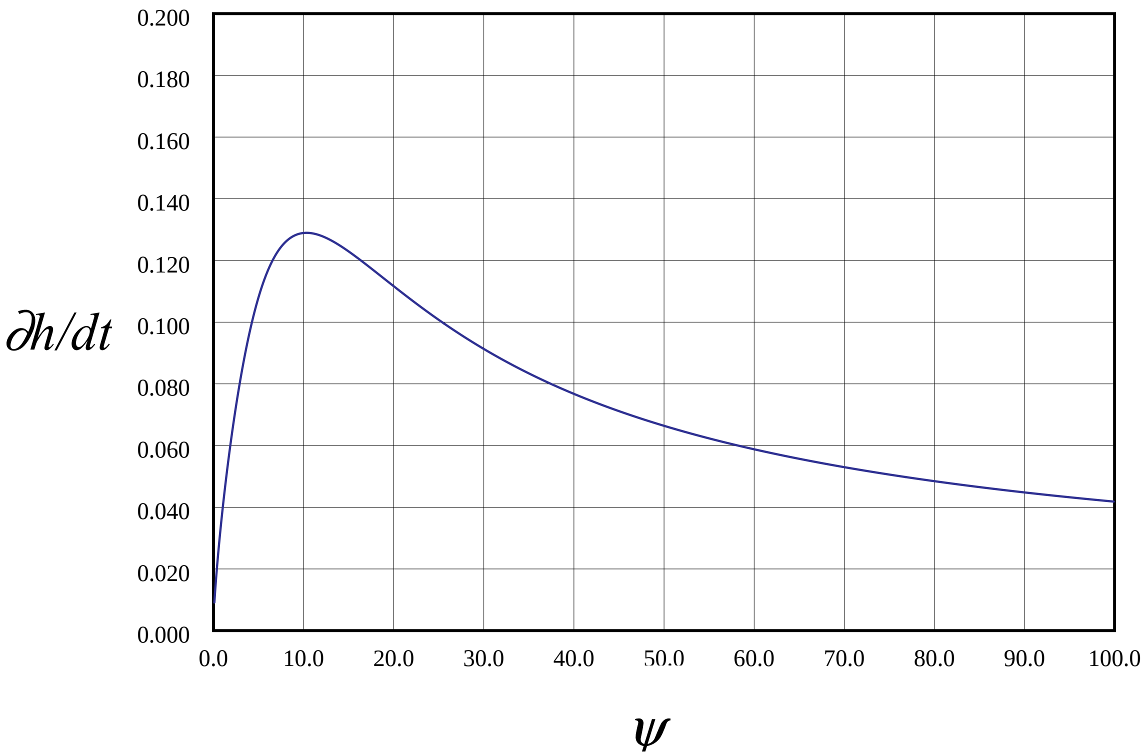
$$\frac{dh}{dt} = \left(\frac{Ag}{C_D}(1-E)\psi \right)^{1/2} h^{1/2}$$

$$\Rightarrow \qquad h = \frac{Ag}{C_D}(1-E)\psi(t-t_0)^2 = \alpha Ag(t-t_0)^2$$

For $C_D = 10$ and $\psi = 1$, $\alpha = 0.025$.

[Full Layzer growth with $\psi = 1$ gives $\alpha = 0.023$.]

Growth rate maximised with $\psi \sim 10$ giving $\alpha \sim 0.103$

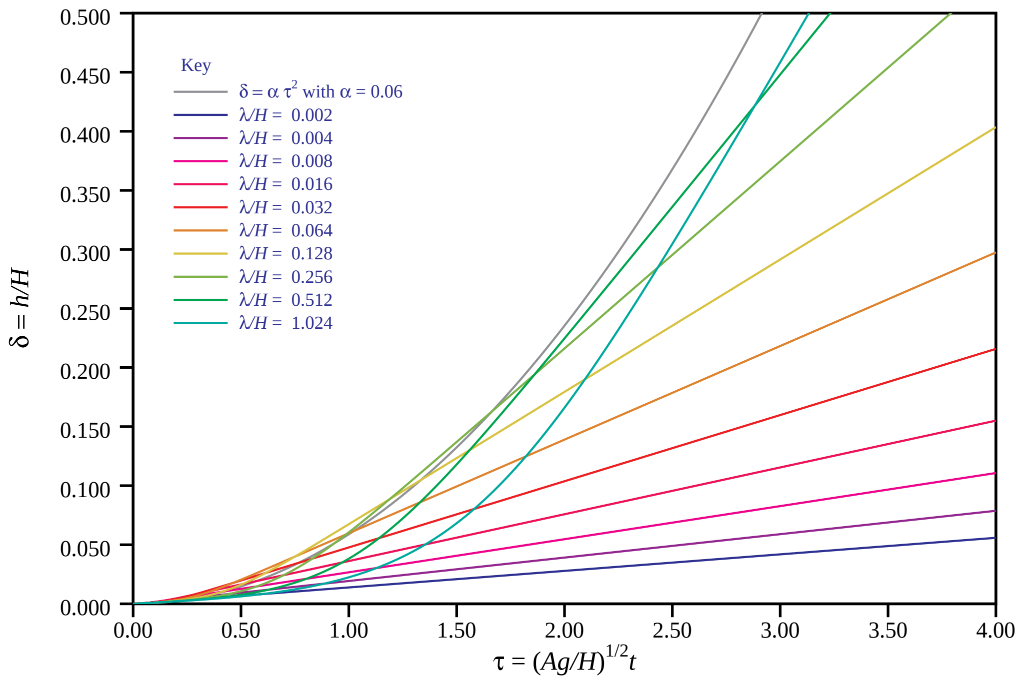


Where do the modes begin? How do they interact?

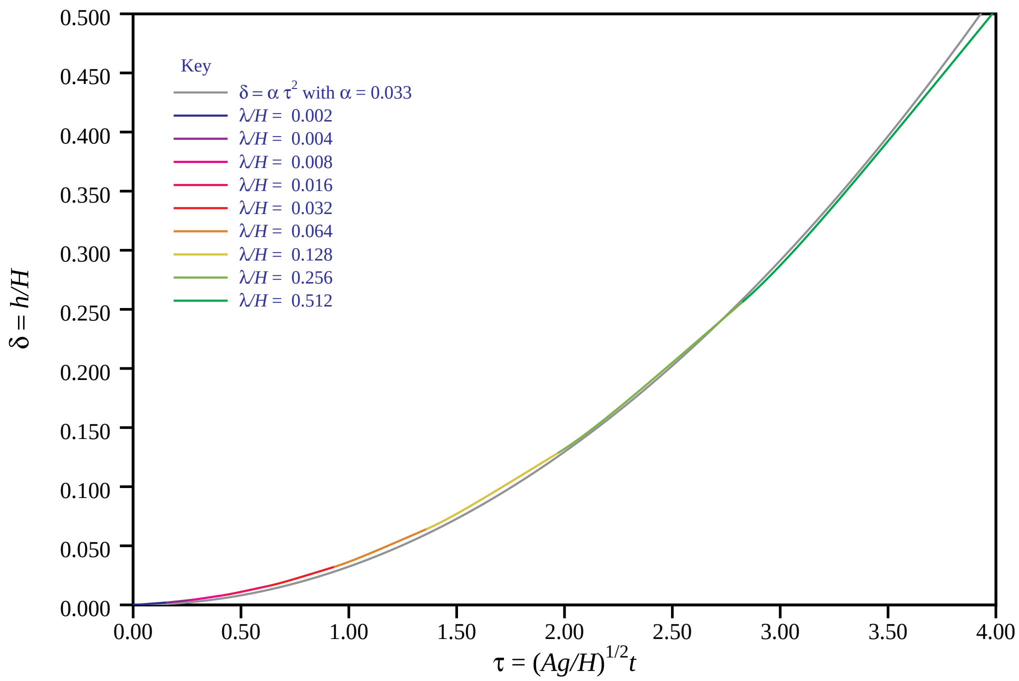
◊ Nonlinear interaction?

◊ Initial perturbation?

If modes independent and equal amplitude:



Instantaneous nonlinear mode halving interaction when $h = \lambda$:



Which is it?

Mixing

See talk by Joanne Holford

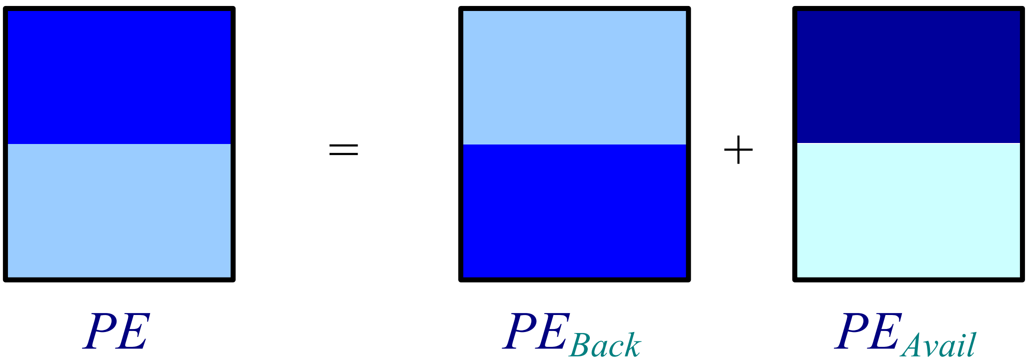
Energy budget *

Can decompose *PE* into **Background PE** and **Available PE**.

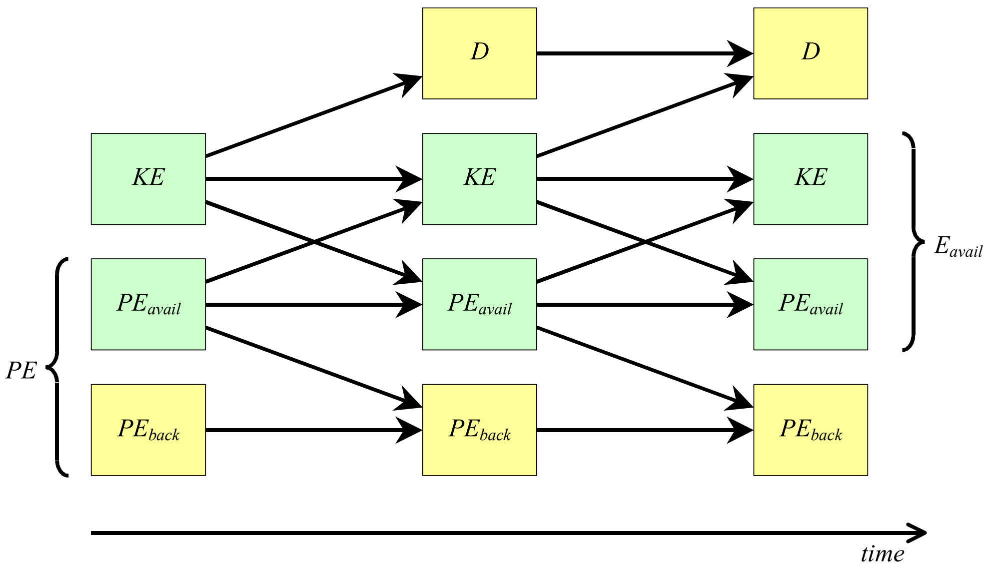
PE_{back} is the minimum energy state that is achieved by adiabatic rearrangement of fluid parcels.

Mixing increases PE_{back} – it cannot decrease it!

PE_{avail} is the component of *PE* that can be converted into *KE*, heat (through dissipation) and, if mixing occurs, into PE_{back}.



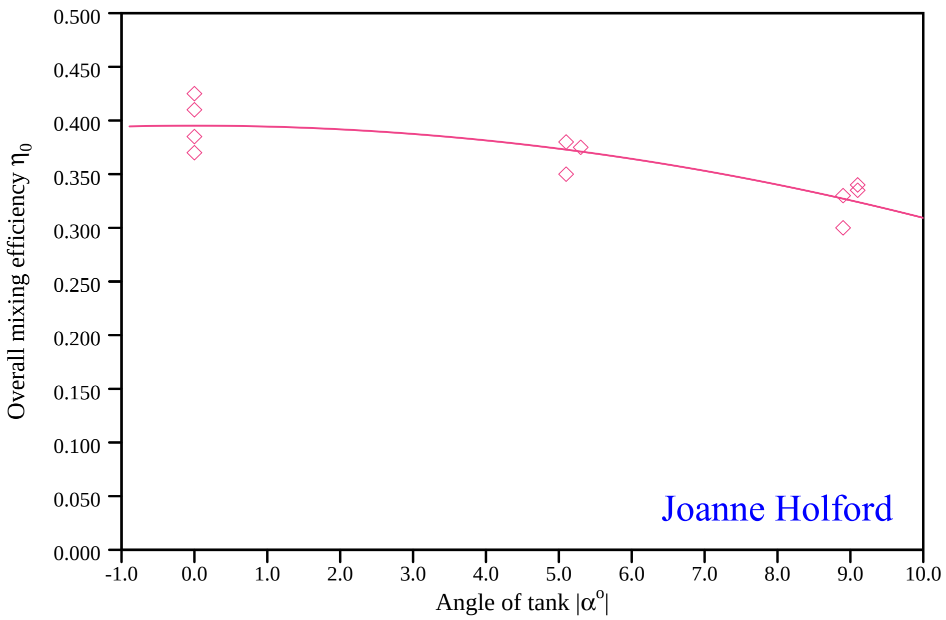
In the absence of external work:



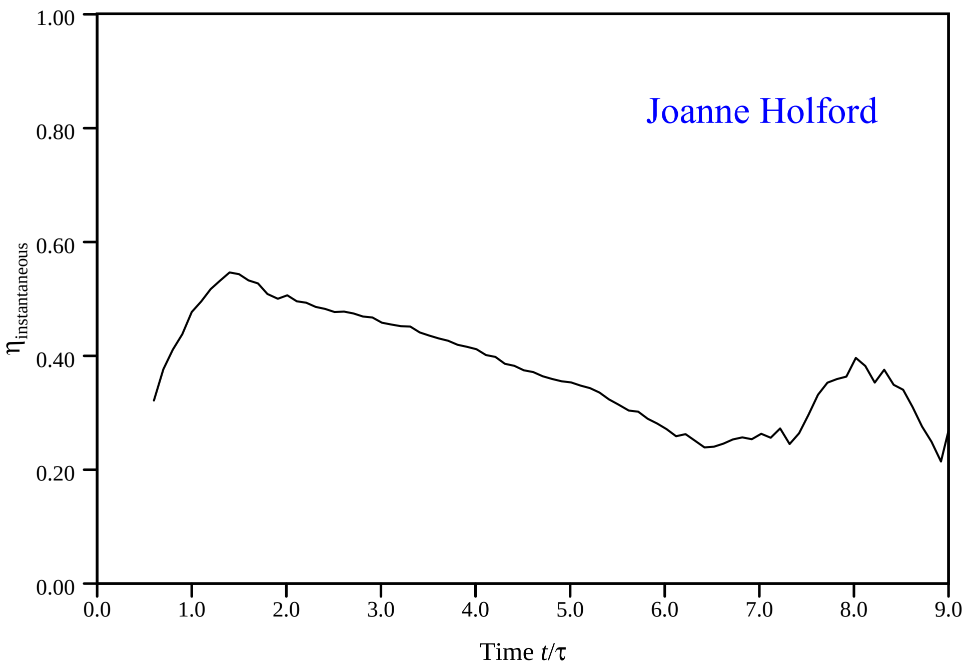
Mixing efficiency

*

$$\eta_{Integral} = -\frac{\Delta E_{back}}{\Delta E_{avail}} = \frac{\Delta PE_{Back}}{\Delta PE_{Back} + \int \varepsilon \, dt} = \frac{\Delta PE_{Back}}{-\left(\Delta KE + \Delta PE_{Avail}\right)}$$



$$\eta_{instantaneous} = \frac{\delta PE_{Back}}{-\delta E_{Avail}} = \frac{\delta PE_{Back}}{\delta KE + \delta PE_{Avail}}$$



Thermal

Entrainment into a thermal

$$\frac{dV}{dt} = \beta w A$$

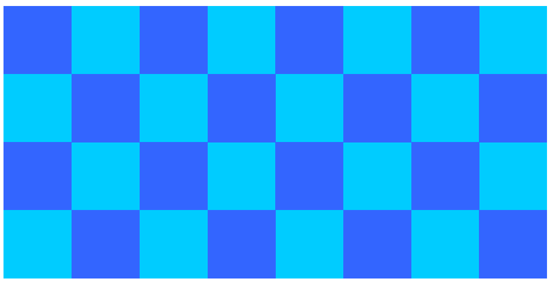
…

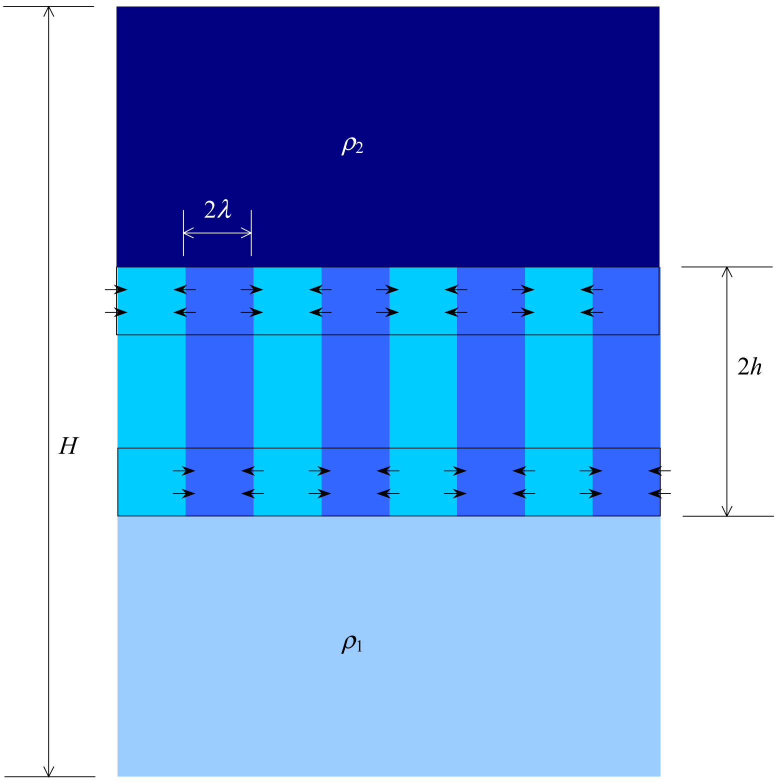
$$\beta = 0.18.$$

Energetics of a thermal

Mixing efficiency not well defined: depends on size of domain!

Rayleigh-Taylor





$$h = \alpha A g t^2 \qquad \delta = \alpha \tau^2$$

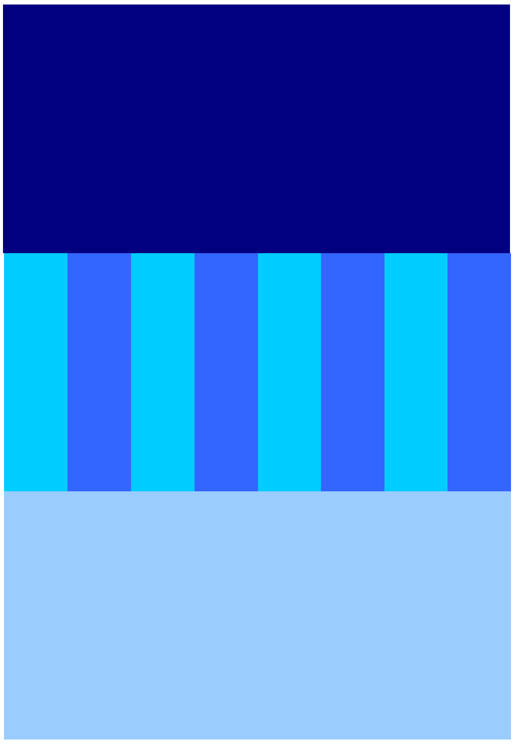
$$w = 2\alpha A g t \qquad \omega = 2\alpha\tau$$

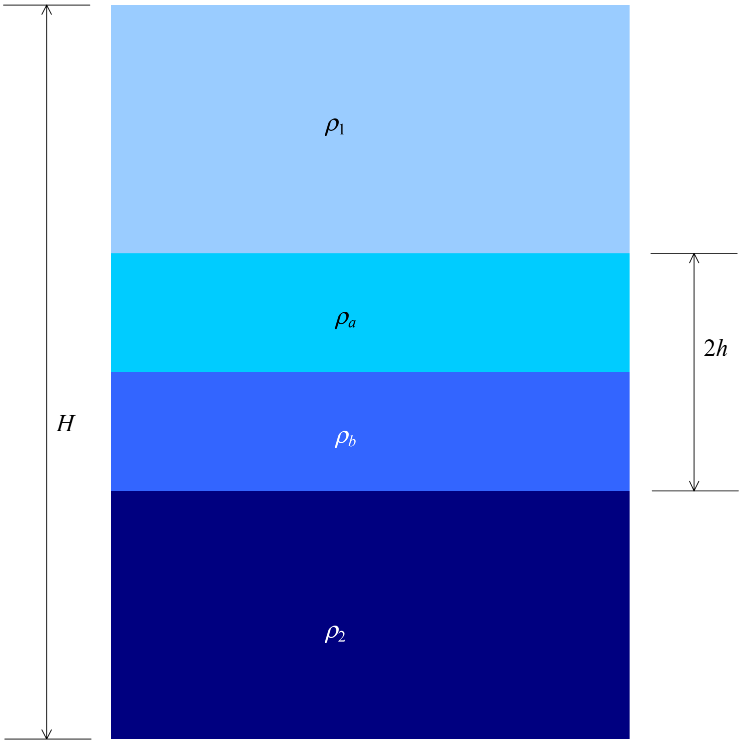
$$V = 2L^2 h$$

Total potential energy

$$PE^*_{Total} = \frac{PE_{Total}}{PE_0} = 1 - 4\alpha^2\tau^4$$

Background potential energy





Changes due to entrainment between counter-flowing streams.

Invoke entrainment hypothesis: $u_e = \beta w$

Area of entrainment independent of h

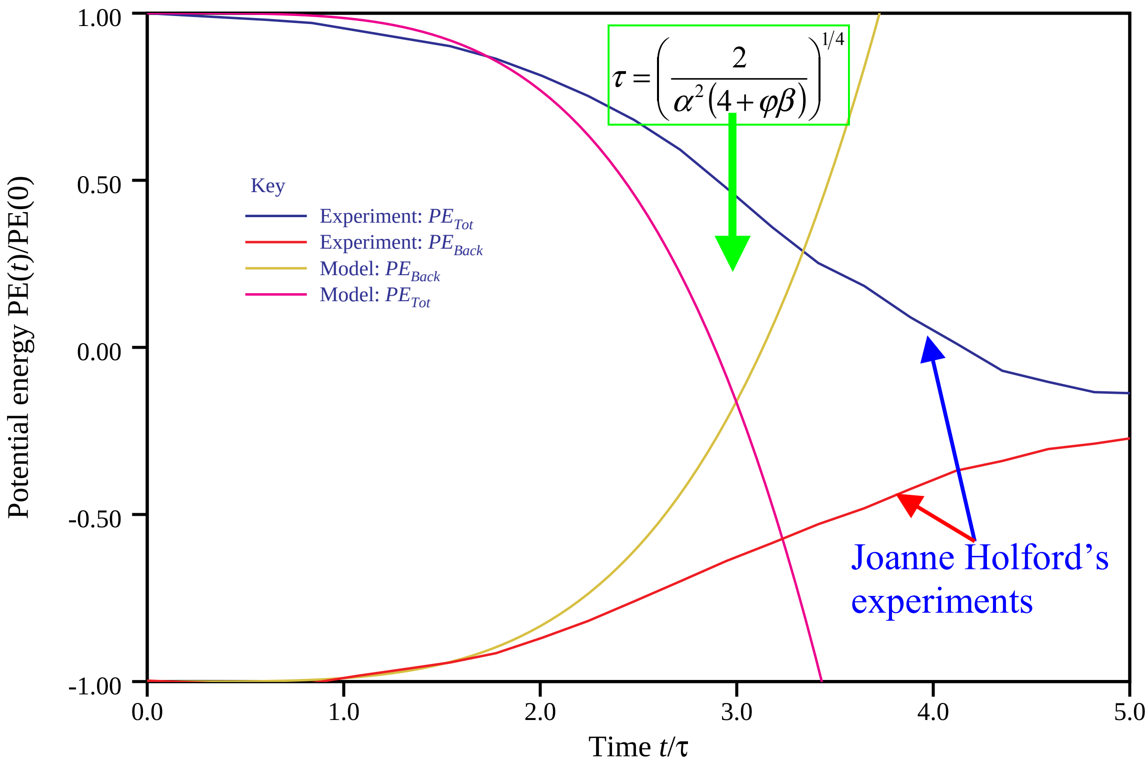
$\Leftrightarrow$ depth of entrainment comparable with λ

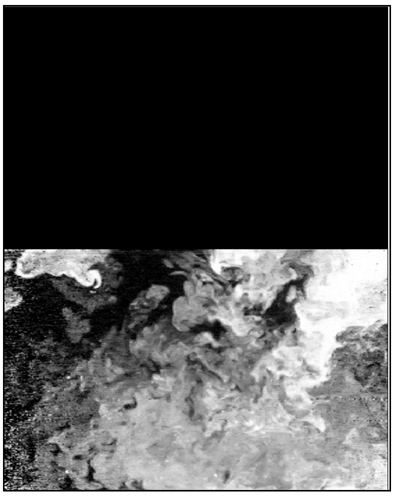
$\Rightarrow$ *entraining area* $= \varphi \times$ *plan area.*

$$PE^*_{Back} = -\left(1 - \varphi\beta\alpha^2\tau^4\right)$$

Available potential energy

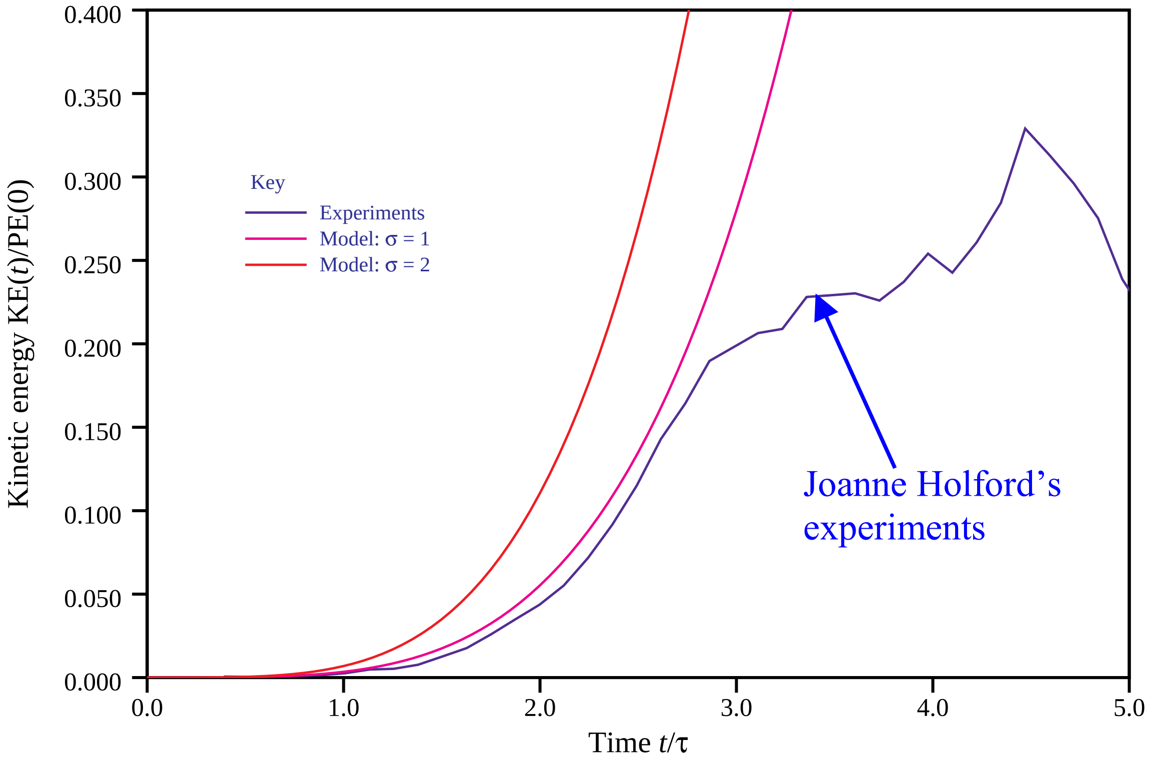
$$PE^*_{Avail} = PE^*_{Tot} - PE^*_{Back} = 2 - (4 + \varphi\beta)\alpha^2\tau^4$$





Kinetic energy

$$KE^* = 16\sigma\alpha^3\tau^4$$



Available energy changing

$$\frac{dE^*_{Avail}}{d\tau} = \frac{dKE^*}{d\tau} + \frac{dPE^*_{Avail}}{d\tau}$$
$$= -4(4 + \varphi\beta - 16\sigma\alpha)\alpha^2\tau^3$$

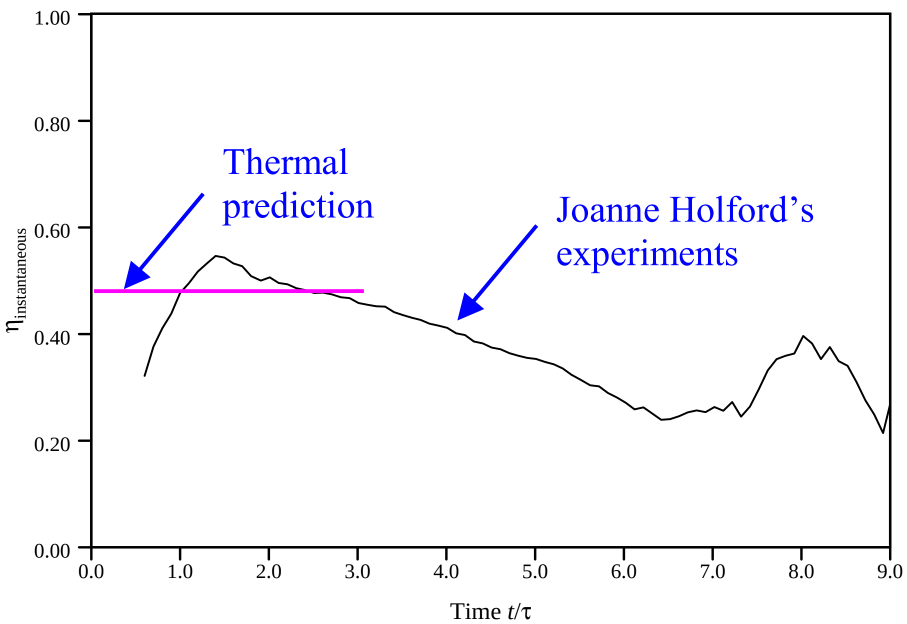
Hence, energy is lost whenever $\alpha < ¼$ (for $\beta = 0$, $\sigma = 1$).

Instantaneous mixing efficiency

$$\eta_{Inst} = -\frac{\dfrac{dPE^*_{Back}}{d\tau}}{\dfrac{dPE^*_{Avail}}{d\tau} + \dfrac{dKE^*}{d\tau}}$$

$$= \frac{\varphi\beta}{4 + \varphi\beta - 16\sigma\alpha}$$

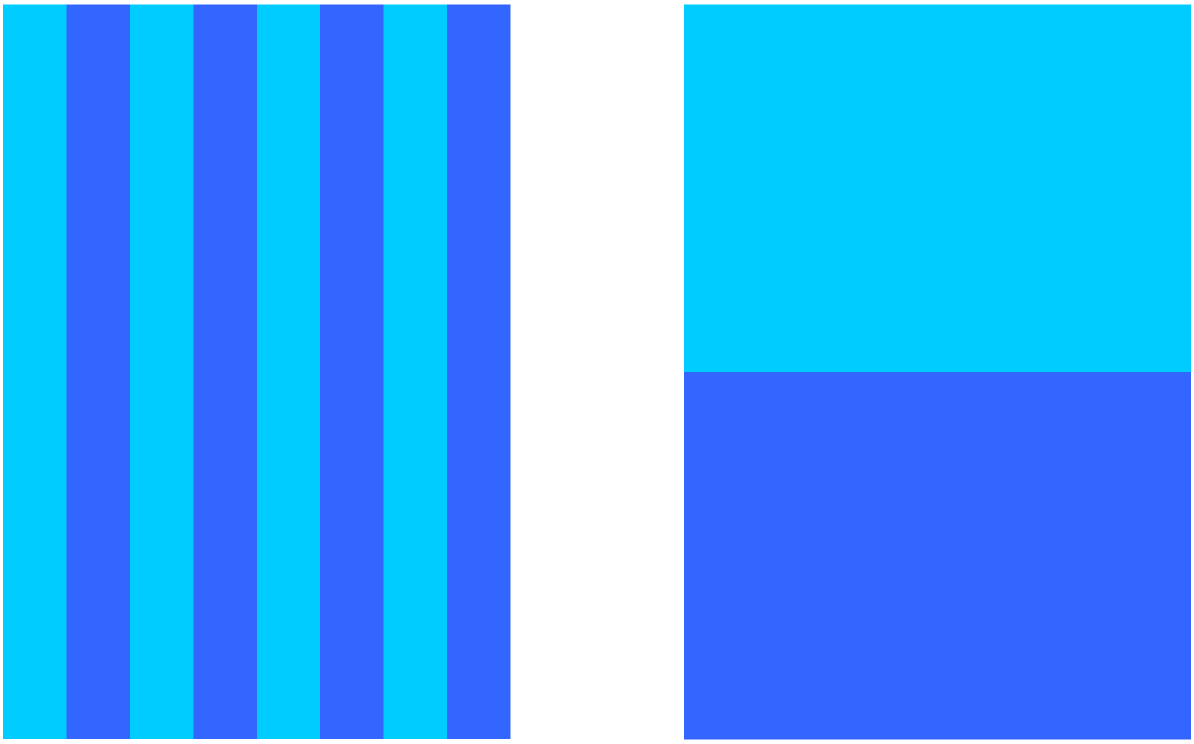
So for $\varphi = 16$, $\beta = 0.18$, $\sigma = 1$, and $\alpha = 0.06$, then $\eta_{Inst} = 0.49$.



Integral mixing efficiency

If there no mixing after reaching the bottom…

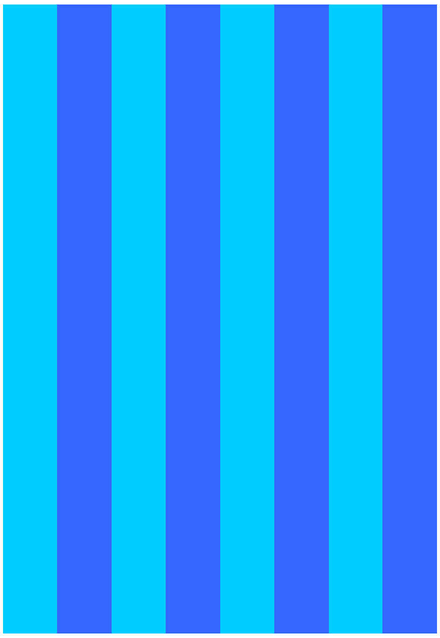
$$\eta_{Integral} = \frac{PE_{Back}^{(bot)} - PE_{Back}^{(0)}}{PE_{Avail}^{(0)}}$$

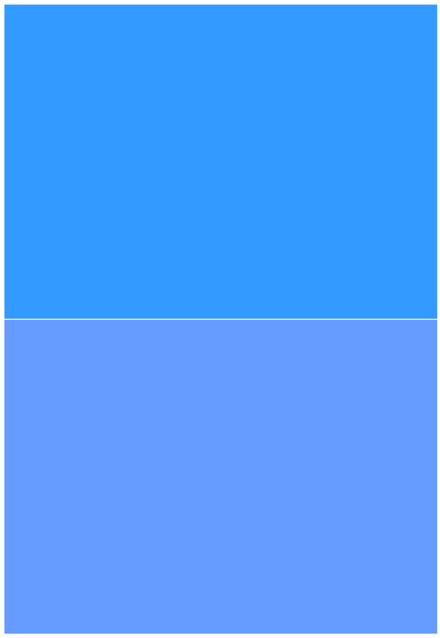


$$\eta_{Integral} = \frac{1}{8}\varphi\beta$$

For $\varphi = 16$ and $\beta = 0.18$, then $\eta_{Integral} = 0.36$.

If there is mixing after reaching the bottom…



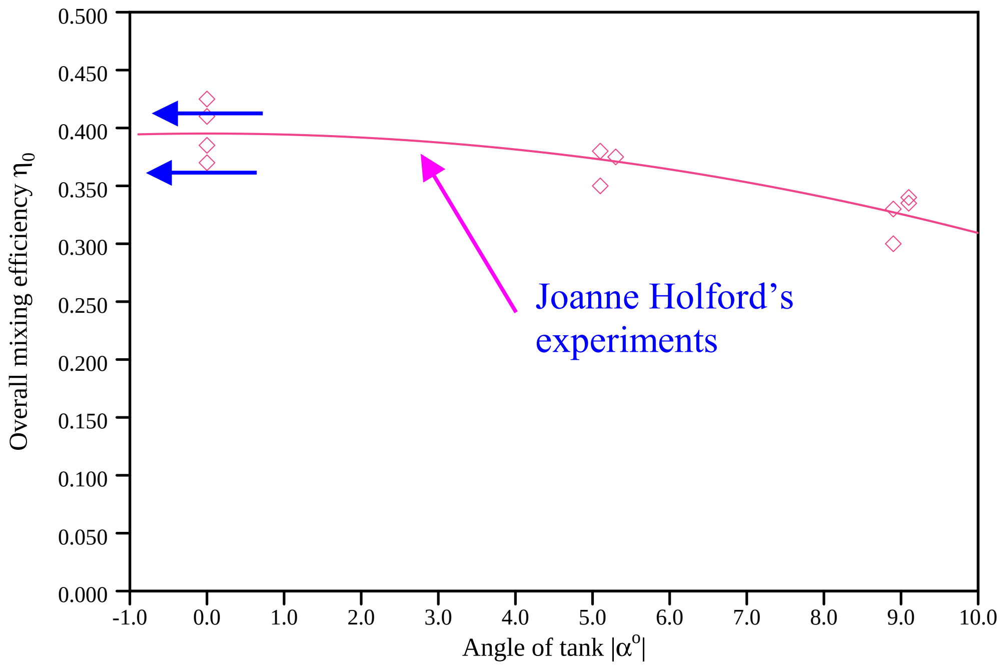


$$E^{*(bot)}_{Avail} = \left(1 + 4\sigma\alpha - \frac{1}{4}\varphi\beta\right)$$

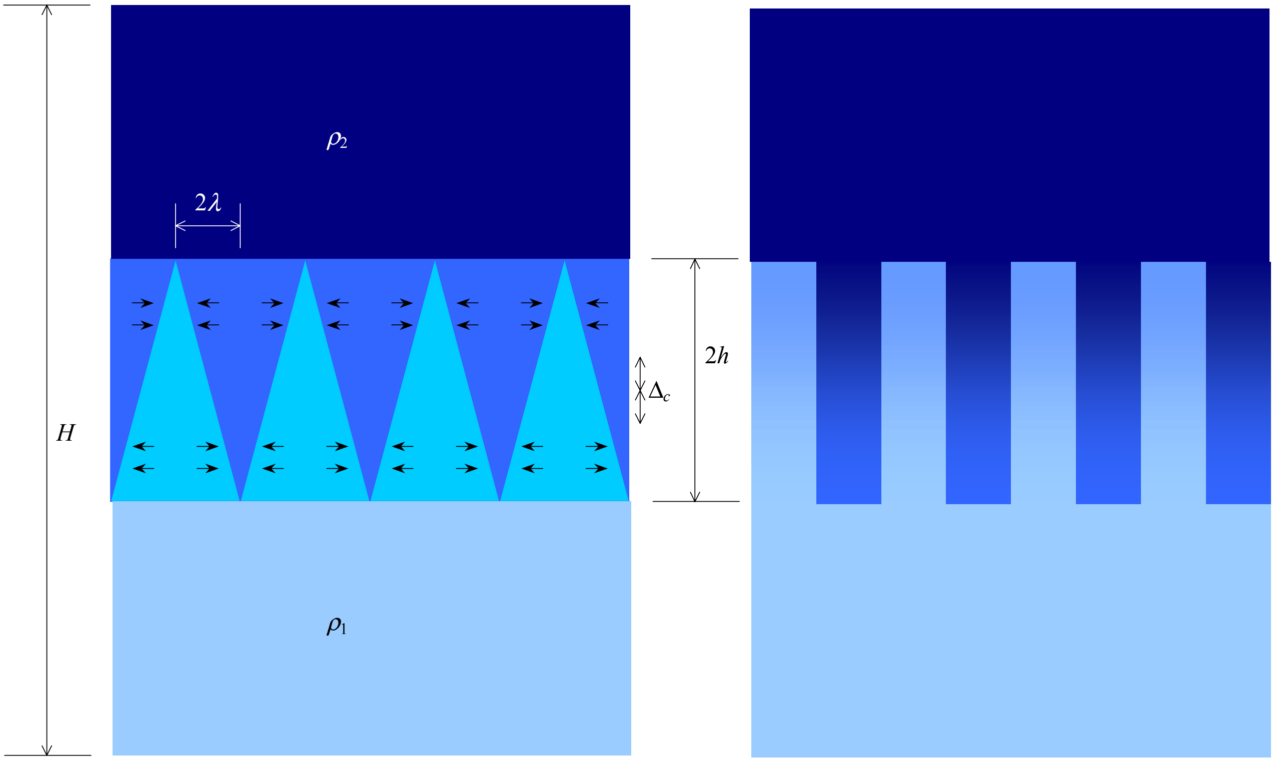
If $\Delta E^{(After\,bot)}_{Back} = \eta_{stab} E^{(bot)}_{Avail}$, then

$$\eta_{Integral} = \frac{1}{8}\varphi\beta + \frac{1}{2}\eta_{stab}\left(1 + 4\sigma\alpha - \frac{1}{4}\varphi\beta\right)$$

For $\eta_{stab} = 0.2$, then $\eta_{Integral} = 0.41$.



Extensions *



Let Δ_c be the fractional displacement of the centroid of the *bubble* from $z = 0$.

→

$$\eta_{Inst} = -\frac{\delta PE^*_{Back}}{\delta PE^*_{Avail} + \delta KE^*}$$

$$= \frac{\varphi\beta}{4 + \varphi\beta - 4(4 - \varphi\beta)\Delta_c - 16\sigma\alpha}$$

Pyramid ($\Delta_c = 1/4$): $\eta_{Inst} = 0.6$.

Parabolic ($\Delta_c = 1/6$): $\eta_{Inst} = 0.56$.

(gives linear mean concentration)

How can we avoid having to specify C_D?

Shell model

GOY model (Gledzer–Ohkitani–Yamada):

$$\frac{dU_n}{dt} = \left(ak_n U^*_{n+1} U^*_{n+2} + bk_{n-1} U^*_{n-1} U^*_{n+1} + ck_{n-2} U^*_{n-1} U^*_{n-2}\right) - \nu k_n^2 U_n + F_n$$

with $k_n = \beta^n k_0$, $a = 1$, $b = -\varepsilon$ and $c = -1 + \varepsilon$.

In Rayleigh-Taylor instability, energy input at all scales.

$$\frac{dU_n}{dt} = \left(k_n U_{n+1} U_{n+2} - \varepsilon\, k_{n-1} U_{n-1} U_{n+1} - (1-\varepsilon) k_{n-2} U_{n-1} U_{n-2}\right) - \nu k_n^2 U_n + F_n$$

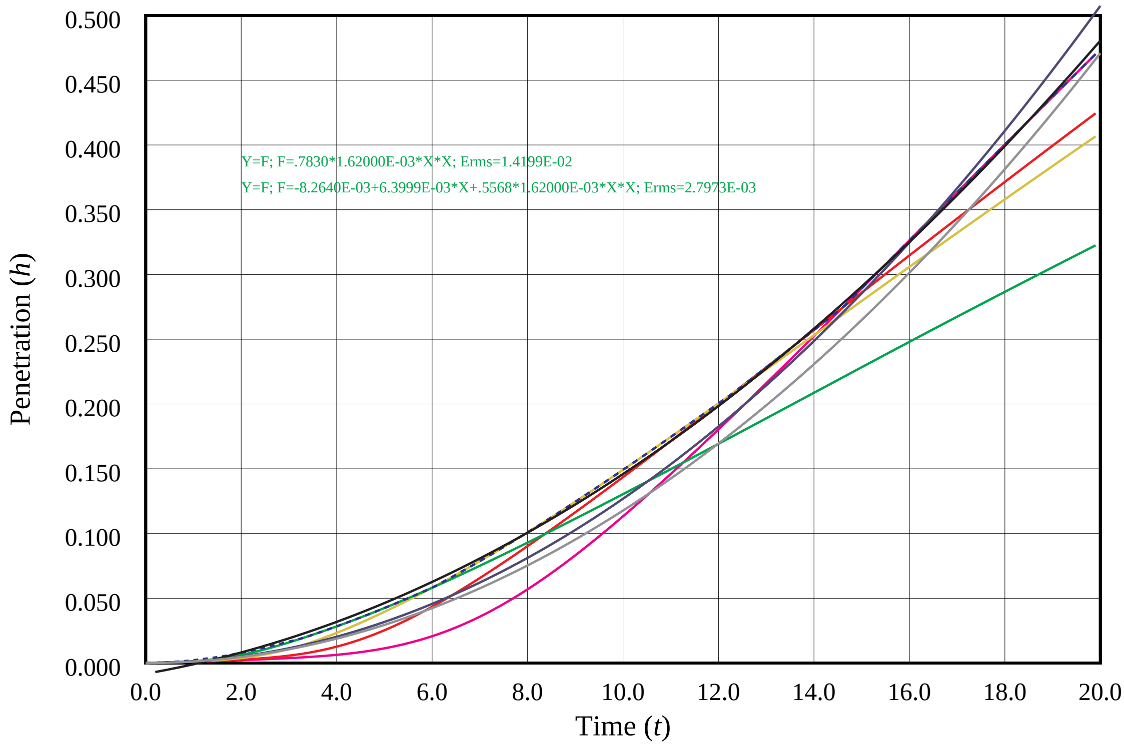
Recall Layzer model: $(2+E)\dfrac{dw}{dt} = Ag(1-E) - C_D \dfrac{w^2}{\lambda}$

Hence $F_n = A_n g \dfrac{1-E_n}{2+E_n}$, where

$$E_n = \exp\left(-\frac{6\pi\, h_n}{\lambda_n}\right) \quad \text{and} \quad A_n = A\, h_n / h.$$

The mode penetrations h_n and total penetration h are obtained from

$$\frac{dh_n}{dt} = U_n \qquad \text{and} \qquad h = \max_n h_n .$$



- ◊ Approximate quadratic growth
- ◊ Coefficient depends on initial spectrum
- ◊ Possible to replicate $\alpha \sim 0.06$

Conclusions

General

- Initial conditions are important for gross features
- Internal details relatively insensitive to initial conditions
- Appropriate modelling of initial conditions gives close agreement

Thermals model

- Single-mode growth rate consistent with isolated thermal
- Simple model for transfer between modes replicates t^2 growth
- Mixing efficiency consistent with thermal entrainment

Shell model

- Baroclinic input at all scales
- Very simple model replicates t^2 growth
- Growth rate sensitive to initial spectrum

An explanation?

No, but it helps.