Nonlinear asymptotic solutions to Rayleigh-Taylor and Richtmyer-Meshkov problems for fluids with a finite density contrast

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LIGHT FLUID ACCELERATES HEAVY FLUID

INSTABILITY of the interface

TURBULENT MIXING of the fluids

Rayleigh-Taylor instability sustained acceleration (gravity)

Richtmyer-Meshkov instability impulsive acceleration (shock)

Fundamental issues:

- the cascades of energy
- the dynamics of the large-scale coherent structure

<u>Coherent structure</u>: a spatially periodic array of bubble and spikes

basic scales: period λ, gravity g (RTI), initial velocity v_0 (RMI)

time scale $\tau \sim \sqrt{\lambda / Ag}$ (RTI) $\tau \sim \lambda / A|v_0|$ (RMI)

Atwood number $A = (\rho_h - \rho_l)/(\rho_h + \rho_l)$

density ratio is a determining physical factor in RTI/RMI dynamics

Heuristic models

RTI, traditional approach $g \to Ag$ Sharp 1984

RMI, buoyancy-drag model $g \to (2A/(1+A))g$ for bubbles

Shvarts 1995 $g \to (2A/(1-A))g$ for spikes

more formal theoretical approach and a systematic study

INTERFACE	**active** regions	**passive** regions
	small scales	**large** scales
	intensive vorticity	simply advected

large-scale coherent motion — scalar fields

time t coordinates (x, y, z) free surface $z^*(x, y, t)$

$z^*(x, y, t) < z < +\infty$ density $\rho = \rho_h$ velocity $\boldsymbol{v} = \boldsymbol{v}_h$

$z^*(x, y, t) > z > -\infty$ density $\rho = \rho_l$ velocity $\boldsymbol{v} = \boldsymbol{v}_l$

scalar function $\theta(x, y, z, t) = z^*(x, y, t) - z$

Conservation laws:

$$\nabla \cdot \boldsymbol{v} = 0$$

momentum $\left(\dot{\boldsymbol{v}}_h + (\boldsymbol{v}_h \nabla)\boldsymbol{v}_h - \boldsymbol{g}\right)\rho_h\big|_{\theta=0} = \left(\dot{\boldsymbol{v}}_l + (\boldsymbol{v}_l \nabla)\boldsymbol{v}_l - \boldsymbol{g}\right)\rho_l\big|_{\theta=0}$

mass $\left(\dot{\theta} + \boldsymbol{v}_h \nabla\theta\right)\rho_h\big|_{\theta=0} = \left(\dot{\theta} + \boldsymbol{v}_l \nabla\theta\right)\rho_l\big|_{\theta=0}$

no mass flux $\left(\dot{\theta} + \boldsymbol{v}_h \nabla\theta\right)\rho_h\big|_{\theta=0} = \left(\dot{\theta} + \boldsymbol{v}_l \nabla\theta\right)\rho_l\big|_{\theta=0} = 0$

boundary conditions $\boldsymbol{v}_h\big|_{z=+\infty} = \boldsymbol{v}_l\big|_{z=-\infty} = 0$

$g \geq 0$ - instability: RTI $g > 0$, RMI $g = 0$

$0 < A \leq 1$ no significant energy cascade potential approximation

$$\boldsymbol{v}_{h(l)} = \nabla\Phi_{h(l)}$$

- Fourier expansion

$$\Phi = \sum_{n=0}^{\infty}\Phi_n(t)\left(\frac{1}{\gamma_n}\exp\left(-\gamma_n(z - z_0(t)) + i\sum_j \boldsymbol{k}_j\boldsymbol{r}n_j\right) + c.c\right)$$

$$\Phi_h \Rightarrow \{\Phi_n\}, \quad \Phi_l \Rightarrow \{\tilde{\Phi}_n\}, \quad \boldsymbol{r} = (x, y), \qquad \boldsymbol{k}\text{-wave-vectors}$$

- Spatial expansion at a highly symmetric point of the interface

$$x \approx 0, y \approx 0, z \approx z_0(t) \qquad z^*(x, y, t) - z_0(t) = \sum_{i+j=N=1}^{\infty}\zeta_{ij}(t)x^{2i}y^{2j}$$

Conservation laws $x^{2i}y^{2j}$ $i + j = N = 1,2,\ldots\infty$

are reduced to dynamical system of ODE in terms of

surface variables $\xi_{ij}(t)$ and

moments $M_n(t) = \sum_{m=0}^{\infty}\Phi_m(t)(km)^n$, $\tilde{M}_n(t) = \sum_{m=0}^{\infty}\tilde{\Phi}_m(t)(km)^n$

with $v = \partial z_0/\partial t = -M_0 = \tilde{M}_0$

- Local dynamics at any time t; the length scale λ is unchanged
- Multiple harmonics analysis, $m = 0,1,2,\ldots\infty$
- Desired accuracy, $i + j = N = 1,2,\ldots\infty$
- 3D flows with general type of symmetry and 2D flows

2D Rayleigh-Taylor and Richtmyer-Meshkov instabilities

N=1: $z^* - z_0 \approx \zeta_1(t)x^2$ principal curvature $\zeta_1(t)$

$$\left(\dot{M}_1/2 + \zeta_1\dot{M}_0 - M_1^2/2 + \zeta_1 g\right)\rho_h = \left(\dot{\tilde{M}}_1/2 - \zeta_1\dot{\tilde{M}}_0 - \tilde{M}_1^2/2 + \zeta_1 g\right)\rho_l$$

$$\left(\dot{\zeta}_1 - 3\zeta_1 M_1 - M_2/2\right)\rho_h = \left(\dot{\zeta}_1 - 3\zeta_1\tilde{M}_1 + \tilde{M}_2/2\right)\rho_l$$

no mass flux: $\left(\dot{\zeta}_1 - 3\zeta_1 M_1 - M_2/2\right)\rho_h = \left(\dot{\zeta}_1 - 3\zeta_1\tilde{M}_1 + \tilde{M}_2/2\right)\rho_l = 0$

Layzer-type expansion

amplitudes Φ_1 and $\tilde{\Phi}_1$

Regular asymptotic solutions

A=1 Layzer 1955

Rayleigh-Taylor bubbles time scale $\tau = 1/\sqrt{Agk}$

$t \ll \tau$ $v, \zeta_1 \sim \exp(t/\tau)$

$t \gg \tau$ $\zeta_1 = \zeta_L = -Ak/6$ $v_L = \sqrt{Ag/3k}$

re-scaling Layzer-type steady bubble A=1

Richtmyer-Meshkov bubbles time scale $\tau = 1/Akv_0$

$t << \tau$ $\zeta_1 = -(k/A)(t/\tau)$ $v - v_0 = -v_0(t/\tau)$

$t >> \tau$ $\zeta_1 = \zeta_L = -Ak/6$ $v_L = (1 - A^2/3)/Akt$

Singular asymptotic solutions

A=1 Zhang 1998, Abarzhi 2000

Rayleigh-Taylor spikes time scale $\tau = 1/\sqrt{Agk}$

$t << \tau$ $v, \zeta_1 \sim \exp(t/\tau)$

$t >> \tau$ $\zeta_1 \approx k \exp\left((3/2)(t/A\tau)^2\right)$ $v \approx -g\, t$

Richtmyer-Meshkov spikes time scale $\tau = 1/Akv_0$

$t << \tau$ $\zeta_1 = -(k/A)(t/\tau)$ $v - v_0 = -v_0(t/\tau)$

$t >> \tau$ $\zeta_1 \approx k \exp\left(C\left(t/\tau A^2\right)\right)$ $v - Cv_0 \approx \exp\left(C\left(t/\tau A^2\right)\right)$

finite – time singularities:

Baker, Meiron 1980s, Moore 1980s, Tanveer 1990s

!!! **Layzer-type expansion requires**

MASS FLUX through the interface

NON-LINEAR REGULAR ASYMPTOTIC SOLUTION

NO MASS FLUX through the INTERFACE

1. Non-linearity is non-local
2. Interplay of harmonics   bubble shape   singularities
3. Multiple harmonics analysis
4. The bubble shape is free   principal curvature
5. Family of regular asymptotic solutions with no mass flux through the interface
6. The fastest solution in the Family   physically dominant
7. Family of asymptotic solutions at *A*=1 in 2D RTI (Garabedian) and 3D RTI and 3D/2D RMI (Abarzhi)

Family of regular asymptotic solutions

Rayleigh-Taylor bubbles, 3D/2D:

$t >> \tau$ $v = v(\zeta_1, A)$

the fastest solution in the family $\zeta_1 = \zeta_A$ $v = v_A$

Richtmyer-Meshkov bubbles, 3D/2D

$t >> \tau$ $v = v(\zeta_1, A, t) = L(\zeta_1, A)/t$

the fastest solution in the family $\zeta_1 = \zeta_A$ $v = v_A = L_A/t$

!!!   lowest-order harmonics $\Phi_1, \tilde{\Phi}_1$ are dominant

2D Rayleigh-Taylor bubble

$t >> \tau$ ζ_A v_A

$A \approx 1,\ \zeta_A \approx -(k/6)(1-(1-A)/8),\ v_A \approx \sqrt{g/3k}\,(1-3(1-A)/16)$

$A \approx 0,\ \zeta_A \approx -(k/2)A^{1/3},\qquad v_A \approx (3/2)^{3/2}\sqrt{Ag/3k}$

!!! For $0 < A \le 1$, velocity v_A is quite close (10-15%) to

$v_L = \sqrt{Ag/3k}$ traditional empirical approach

$v_D = \sqrt{2A/(1+A)}\sqrt{g/3k}$ drag model

Bubble curvature is a more sensitive parameter

2D Richtmyer-Meshkov bubble

$t >> \tau$ $\zeta_A = 0$ $v_A = 3/2Akt$

!!! Agreement with multiple harmonic analysis at A=1 (S.A. 2000)

!!! Qualitative agreement with experiments

RM bubbles decelerate RM bubbles flatten

$$v_A/v_L = 3/2\left(1 - A^2/3\right)$$

$$v_A, v_L \sim C/kt \qquad \Delta h \sim C\ln(t/\tau)$$

!!! Bubble curvature is a more sensitive parameter

!!! $A \to 0$ and $(t/\tau) \to \infty$ $v(kt) >> 1$

SIMULATIONS

Front Tracking method FronTier (Glimm, 1988)

- 2D compressible adiabatic Navier-Stokes equation

Euler equations augmented viscous forces and heat flux

- weakly compressible fluids
- contribution of viscous and thermal terms is small to yield a slightly stabilized but nearly inviscid calculations

<u>mesh refinement</u>: 80 x 800, 160 x 1600, 320 x 3200

$A < 0.05$ slow evolution no satisfactorily late-time convergence

$A > 0.85$ certain numerical restrictions late-time dynamics

$$0.3 < A < 0.8$$

Nonlinear regime

Bubble: terminal velocity accompanied by slight oscillations

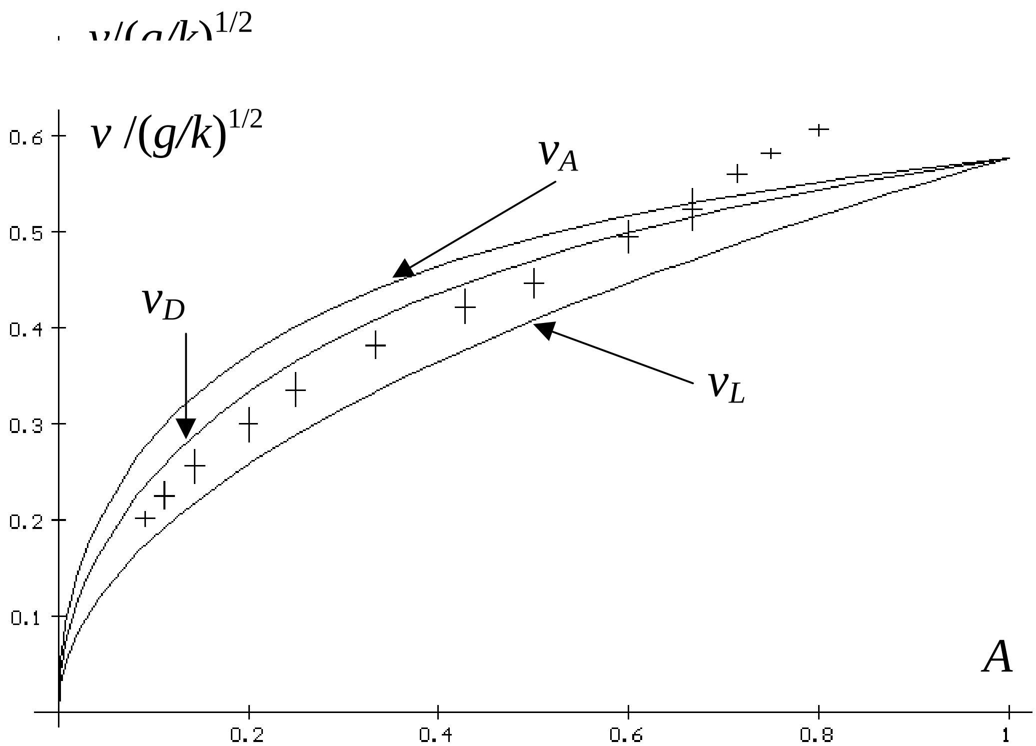
oscillations: small amplitude sensitive to A

$A > 0.7$: terminal velocity

Comparison

numerical data quasi-terminal regime for each A

averaged values **deviations 3-8%**



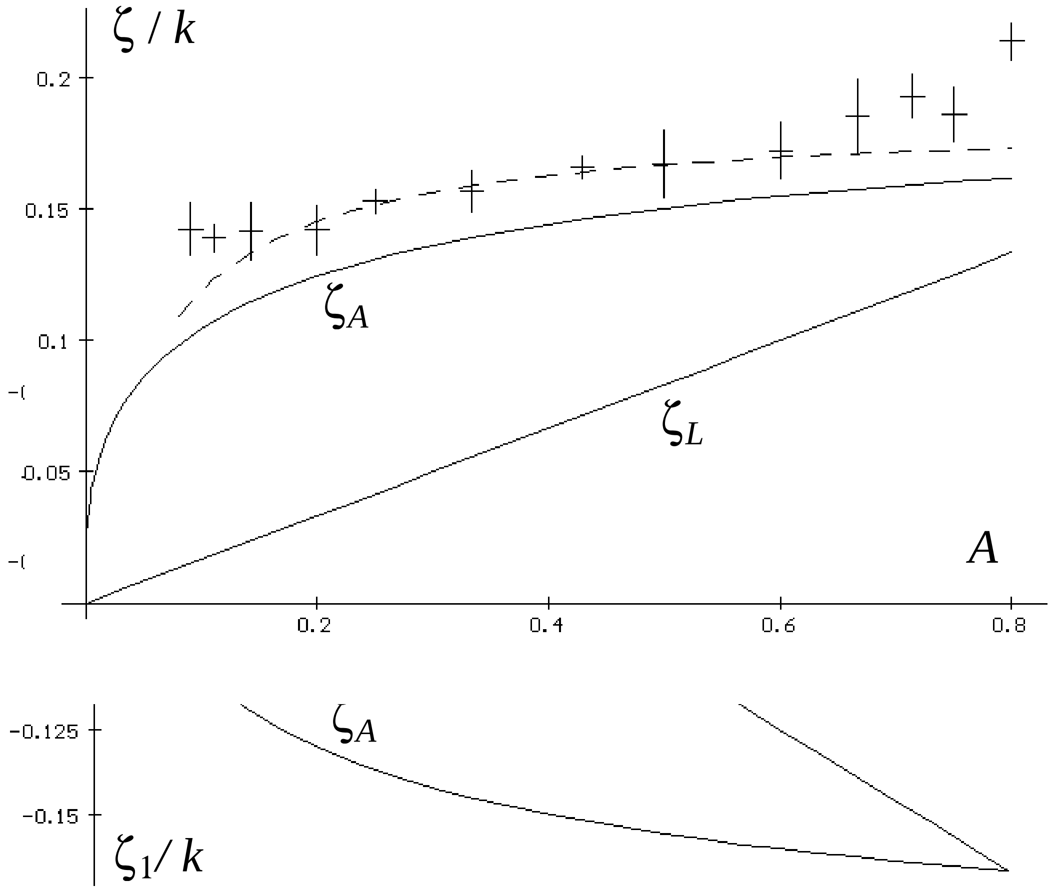
2D RTI

Dependence of the quasi-steady velocity on the Atwood number.

$v_L = \sqrt{Ag/3k}$ is the velocity of the Layzer-type bubble,

$v_D = \sqrt{g/3k}\sqrt{2A/(1+A)}$ corresponds to drag model, and

v_A corresponds to nonlinear solution with no mass flux through the interface.



2D RTI:

Dependence of the curvature of the quasi-steady bubble on the Atwood number. The curvature of the Layzer-type bubble is $\zeta_L = -Ak/6$; the curvature corresponding to the nonlinear solution with no mass flux is ζ_A.

LIMITATIONS:

NON-LINEAR SOLUTIONS are "QUASI-STEADY"

vorticity   energy cascade   time-dependence

$A \to 0$ applicability of the theory

NON-LINEAR SINGULAR ASYMPTOTIC SOLUTIONS

$A \approx 1$ finite-time singularities

$A < 1$ vorticity   energy cascade

small-scale structures   dispersive properties of the flow

ADVANTAGES:

3D Rayleigh-Taylor and Richtmyer-Meshkov instabilities

for fluids with a finite density contrast

!!!   CHAOTIC REGIME

RTI   width of the mixing zone $h \approx \alpha A g t^2$

re-scaling $g \to Ag$ mass flux $\alpha = \alpha(A)$

CONCLUSIONS

1. Rayleigh-Taylor and Richtmyer-Meshkov instabilities for fluids with a finite density contrast in 3D and 2D
2. Analytical solutions for the conservation laws
3. Layzer-type solution in RTI/RMI, re-scaling
4. Layzer-type approach requires mass flux through the interface
5. Approximate nonlinear solution with no mass flux
6. Parameters of the RT and RM bubbles
7. RT bubble is curved, RM bubble is flat
8. The bubble curvature is a more sensitive parameter than the bubble velocity
9. Good quantitative agreement between theory and simulations in RTI
10. Comparison with heuristic models
11. Limitations