# High order methods for 2D Richtmyer-Meshkov Instability

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# Richtmyer-Meshkov Instability

- Shock induced instability of the interface between fluids of different density (Theoretized by Richtmyer and experimented by Meshkov.)
- Growth of the interface amplitude and secondary shear instability promote the onset of turbulence mixing.
- Applications included but not limited to mixing enhancement and inertial confinement fusion (ICF).

## Richtmyer-Meshkov Instability (Cont.)

The two-dimensional Euler equations,

$$\mathbf{Q}_t + \mathbf{F}_x + \mathbf{G}_y = 0.$$

The state vector $\mathbf{Q}$ is

$$\mathbf{Q} = (\rho, \rho u, \rho v, E)^{\mathrm{T}} \ .$$

The inviscid fluxes $\mathbf{F}$ and $\mathbf{G}$ are given by

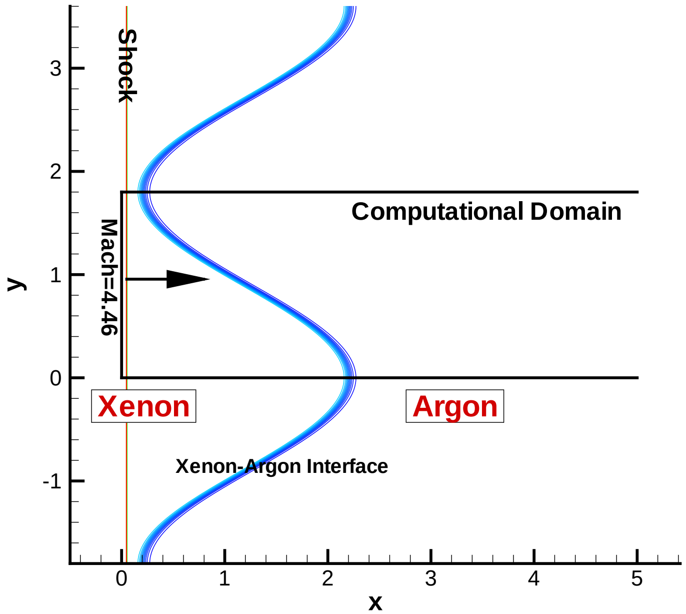
$$\begin{aligned}
\mathbf{F} &= (\rho u, \rho u u + P, \rho u v, (E + P)u)^{\mathrm{T}} , \\
\mathbf{G} &= (\rho v, \rho u v, \rho v v + P, (E + P)v)^{\mathrm{T}} .
\end{aligned}$$

where

$$P = (\gamma - 1)(E - \frac{1}{2}\rho \mathbf{U} \cdot \mathbf{U}) \, .$$

# Richtmyer-Meshkov Instability (Cont.)

## Initial Condition :




- Hugoniot-Rankine condition for the shock
- Pre-Shock Temperature $T = 296\ K$
- Pre-Shock Pressure $P = 0.5\ atm$
- Xenon and Argon density are $\rho_{Xe} = 2.90 \times 10^{-3}\ \frac{g}{cm^3}$ and $\rho_{Ar} = 0.89 \times 10^{-3}\ \frac{g}{cm^3}$ respectively, at half of the normal atmospheric pressure
- Specific heat ratio $\gamma = \frac{5}{3}$
- Atwood number $At = 0.54$
- Mach number $M = 2, 4.46, 8, 10$
- Wave Length $\lambda = 36\ mm$
- Amplitude $a = 1, 2.5, 5, 10, 15\ mm$

# Richtmyer-Meshkov Instability (Cont.)

**Xenon-Argon interface definition :**

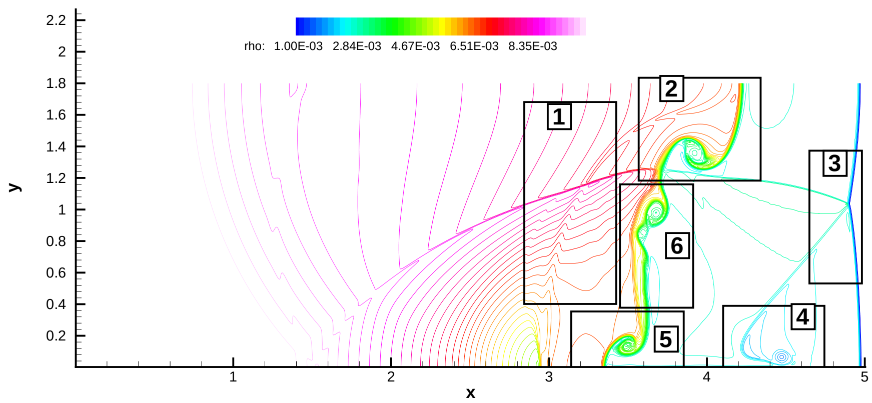
$$S(x,y) = \exp(-\alpha|d|^{\beta}) \qquad 0 < d < 1$$

where

- $$d = \frac{(x_i + a\cos(2\pi y/\lambda) + \delta) - x}{2\delta}$$
- $\delta = a/5$ is the interface thickness,
- $\beta = 8$ is the interface order,
- $\alpha = -\ln \epsilon$ with $\epsilon$ being the machine zero.

# Richtmyer-Meshkov Instability (Cont.)

## Regions of Interest :




1. Reflected shock generated by the shock refraction;
2. The penetration of the heavy (Xe) to light (Ar) fluid forms the Spike;
3. Triple point on the transmitted shock;
4. A small jet and its vortical structure. The Kelvin-Helmholtz instability caused the instability along the contact discontinuity in long time simulation;
5. The penetration of the light (Ar) to heavy (Xe) fluid forms the Bubble;
6. Vortical rollups of the gaseous interface.

## Richtmyer-Meshkov Instability : Algorithms

- Spatial Algorithm : (Spectral)
  1. Combined Chebyshev ($x$) and Fourier collocation method ($y$)
     - Differentiation and Smoothing operations are done via an optimized library PseudoPack (Costa & Don);
     - a 10'th and 9'th order exponential filter used for the differentiation and solution smoothing respectively.
     - Incremental domain size in $x$ as shock moved downstream.
     - Parallelized with OpenMP.
  2. WENO fifth order finite difference scheme (WENO) with Lax-Frederick flux.
  3. Symmetry property in $y$ is utilized to reduce the cost of computation.
- Temporal Algorithm :
  Third order TVD Runge Kutta method (Shu and Osher).

# Spectral Methods - Introduction

The solution of an equation is assumed to be in a space $\mathcal{B}_{\mathcal{N}}$ spanned by smooth basis functions $\{\phi_k\}$.

- For periodic problems

$$\phi_k = e^{ikx}$$

- for non-periodic problems in finite intervals
  - Either

$$\phi_k = T_k(x)$$

    where $T_k$ are the Chebyshev polynomials.
  - Or

$$\phi_k = L_k(x)$$

    where $L_k$ are the Legendre polynomials.

# Spectral Methods - Introduction (Cont.)

The approximation is obtained in one of two ways:

- Galerkin projection:

$$\mathcal{P}_{\mathcal{N}} F(x) = \sum_{k=0}^{N} (F, \phi_k) \phi_k(x)$$

  where

$$(F, \phi_k) = \int \omega(x) F(x) \phi_k(x) \; dx$$

## Spectral Methods - Introduction (Cont.)

- The Pseudo-spectral approximation

$$\mathcal{I}_{\mathcal{N}} F(x) = \sum_{k=0}^{N} (F, \phi_k)_N \phi_k(x)$$

where

$$(F, \phi_k)_N = \sum_{i=0}^{N} \omega_i F(x_i) \phi_k(x_i)$$

- $x_i$ and $\omega_i$ are the Gauss-Lobatto Quadrature nodes and weights respectively.
- Thus

$$\int f(x)\omega(x)dx = \sum_{i=0}^{N} \omega_i f(x_i) \phi_k(x_i)$$

when $f(x)$ is a polynomial or trigonometrical polynomial of degree $2N-1$.

- Alternatively

$$\mathcal{I}_{\mathcal{N}} F(x) = \sum_{i=0}^{N} F(x_i) g_i(x)$$

where

  - $g_i(x)$ is a polynomial of degree $N$ such that $g_i(x_k) = \delta_{i,k}$.

This leads naturally to the formula:

$$\frac{d}{dx} \mathcal{I}_{\mathcal{N}} F(x_k) = \sum_{i=0}^{N} F(x_i) g_i'(x_k)$$

and defines the *Differentiation Matrix*

$$\mathcal{D} = g_i'(x_k)$$

# Approximation Results

$$
\begin{aligned}
||F - \mathcal{P}_N F|| &\leq K \frac{||F||_s}{N^s} \\
||F - \mathcal{I}_N F|| &\leq K \frac{||F||_s}{N^s}
\end{aligned}
$$

# Filtering

- To solve most *nonlinear* PDE, for example Burgers' Equation, some form of dissipation (natural or numerical) is generally required for stability by damping the growth of the high modes.
- Spectral methods is a conservative method.
- Stabilization of the spectral methods requires *Low Pass Filter*.

# Filtering (Cont.)

## Filters in the Transform Space

Force the Fourier/Chebyshev coefficient $a_k$ decays faster.

Replace

$$f_N(x) = \sum_{k=0}^{N} a_k e^{ikx}$$

by

$$\hat{f}_N(x) = \sum_{k=0}^{N} \sigma(\frac{k}{N}) a_k e^{ikx} \quad .$$

# Filtering (Cont.)

## Definition of Filters

A filter of $p$ order $\sigma(x)$ is defined as

$$\begin{aligned} \sigma(0) &= 1 \\ \sigma^l(0) &= 0 \quad 1 \leq l \leq p-1 \\ \sigma^l(1) &= 0 \quad 1 \leq l \leq p-1 \end{aligned}$$

**Theorem** (Vandeven 1989)

$$|f(x) - \hat{f}_N(x)| \leq C_p N^{1-p} d(x)^{1-p} ||f||$$

where $d(x)$ is the distance from the discontinuity.

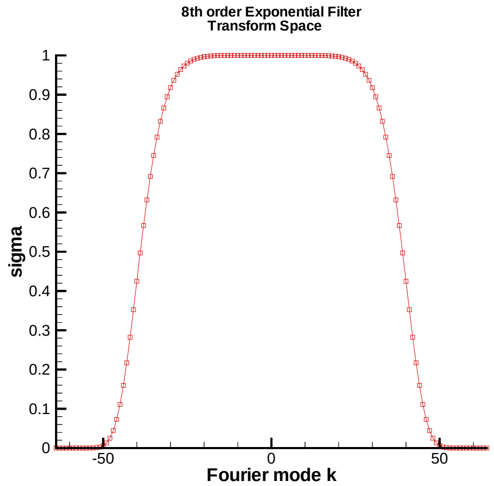
If $p$ is proportional to $N$, accuracy can be recovered *away from the discontinuity*.

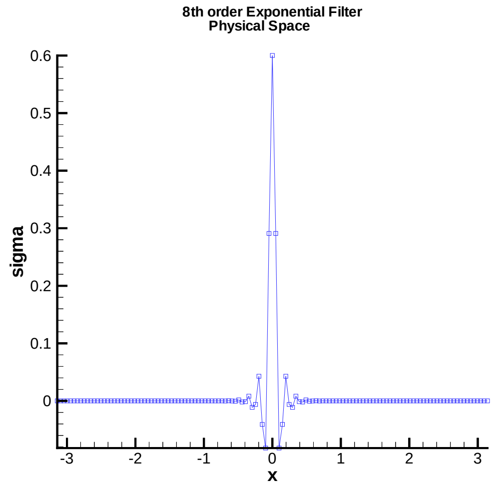
# Filtering (Cont.)

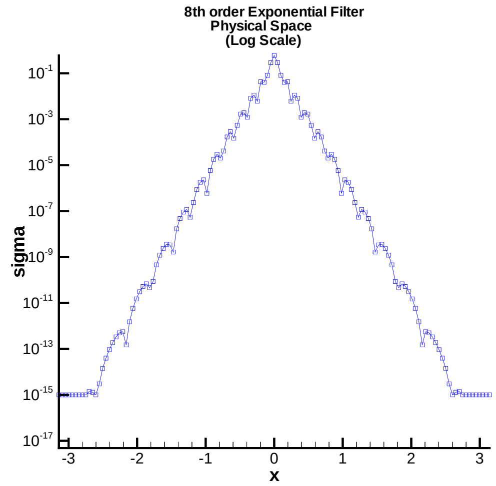
## Exponential Filter

$$\sigma(\frac{k}{N}) = \exp\left(-\alpha\left(\frac{k}{N}\right)^{\gamma}\right)$$

where $\alpha = -\ln \epsilon$, $\epsilon$ is the machine zero.



# Time Stepping Scheme

The algorithm employs the third order TVD Runge-Kutta scheme (Shu & Osher) to solve the system of ODE's produced by the spatial differencing.

It has the form of

$$\begin{aligned}
\vec{U}^1 &= \vec{U}^n + \Delta t L(\vec{U}^n) \\
\vec{U}^2 &= \frac{1}{4}(3\vec{U}^n + \vec{U}^1 + \Delta t L(\vec{U}^1)) \\
\vec{U}^{n+1} &= \frac{1}{3}(\vec{U}^n + 2\vec{U}^2 + 2\Delta t L(\vec{U}^2))\,.
\end{aligned}$$

$L$ is the spatial operator.

$\vec{U}^n$and $\vec{U}^{n+1}$ are the data arrays at the $n$-th and $(n+1)$-th time step, respectively. $\vec{U}^1$and $\vec{U}^2$ are two temporary arrays at the intermediate Runge-Kutta stages.

The scheme is stable for CFL $< 1$.

# Mapping for Chebyshev Method

Consider the scalar hyperbolic equation

$$\begin{aligned} U_t &= U_x \quad -1 \le x \le 1 \\ U(1,t) &= 0\,. \end{aligned}$$

In the semi-discreet form

$$\frac{\partial}{\partial t} U_N(t) = D U_N(t)$$

where $D$ is the Chebyshev differentiation matrix with the first row and column removed.

**<u>Disadvantages :</u>**

1. Roundoff Error grows as $O(N^{2k})$.
2. Restrictive CFL conditions for explicit time stepping scheme with $\Delta t \approx O(N^{-2k})$.

with $k$ being the number of derivatives taken.

# Mapping for Chebyshev Method (Cont.)

## Kosloff-Tal-Ezer Mapping

$$x = g(\xi, \alpha) = \frac{\sin^{-1}(\alpha\xi)}{\sin^{-1}\alpha}$$

where $\xi_j$ and $x_j$ are the original and mapped Chebyshev collocation points respectively.

If $\alpha \in [0, 1]$ is chosen as

$$\alpha = \alpha(N, \epsilon) = \text{sech}\left(\frac{|\ln \epsilon|}{N}\right),$$

then the approximation error is roughly $\epsilon$. Note that $\alpha$ is not a constant but a function of $N$.

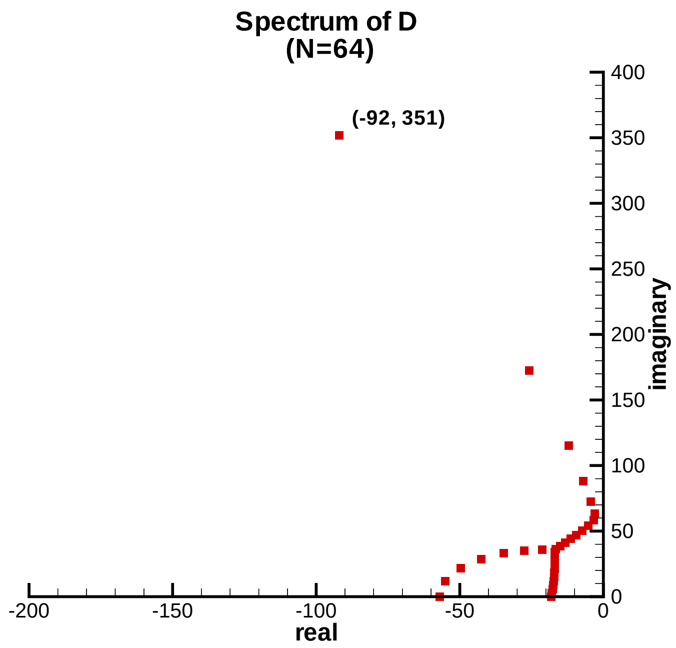
*By choosing $\epsilon$ to be machine zero, the error of the coordinate transformation is essentially guaranteed to be harmless.*

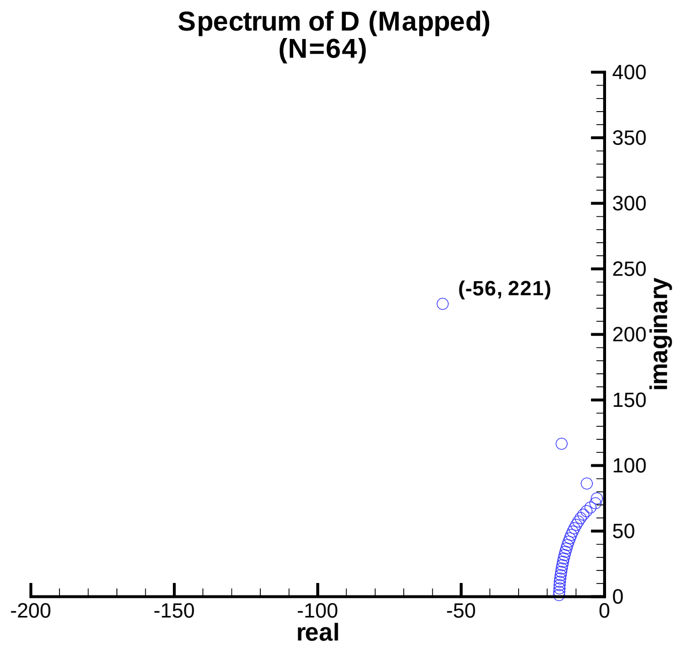
Under the mapping, the differentiation matrix $D$ becomes

$$\mathcal{D} = \mathcal{M}D$$

where $\mathcal{M}$ is a diagonal matrix with element $\mathcal{M}_{ii} = g'(\xi_i, \alpha)^{-1}$.

# Mapping for Chebyshev Method (Cont.)



# Mapping for Chebyshev Method (Cont.)

## Advantages

- Reduction of roundoff error from $O(N^{2k}\epsilon)$ to $O(N^k\epsilon)$.

| $N$ | No Mapping | with Mapping |
|---|---|---|
| 32 | 0.47E-09 | 0.20E-09 |
| 64 | 0.62E-08 | 0.20E-08 |
| 128 | 0.71E-07 | 0.13E-07 |
| 256 | 0.35E-05 | 0.21E-06 |
| 512 | 0.98E-05 | 0.33E-06 |
| 1024 | 0.13E-02 | 0.21E-05 |

Table I: Absolute maximum error for the second derivative of $\sin(2x)$.

# Mapping for Chebyshev Method (Cont.)

- Reduction of the spectral radius of the differentiation matrix from $O(N^{2k})$ to $O(N^k)$ asymptotically for $D$ and $\mathcal{D}$ respectively.

| $N$ | $\lambda(D)$ | Growth Rate | $\lambda(\mathcal{D})$ | Growth Rate |
|---|---|---|---|---|
| 32 | 91.6 | | 80.8 | |
| 64 | 263.8 | 2 | 230.4 | 1.50 |
| 128 | 1452.7 | 2 | 555.4 | 1.27 |
| 256 | 5808.4 | 2 | 1219.1 | 1.13 |
| 512 | 23231.3 | 2 | 2553.5 | 1.07 |
| 1024 | 92922.8 | 2 | 5225.8 | 1.03 |

Table II: The spectral radius $\lambda$ of $D$ and $\mathcal{D}$.

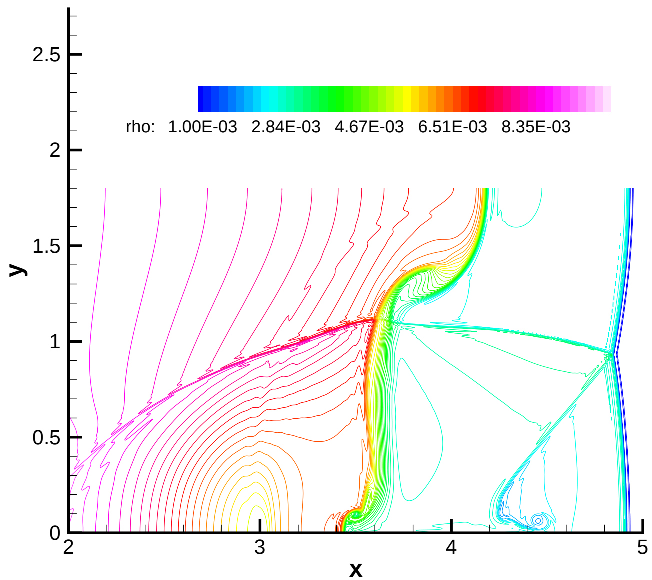
$k$ is the order of differentiation.

# Results

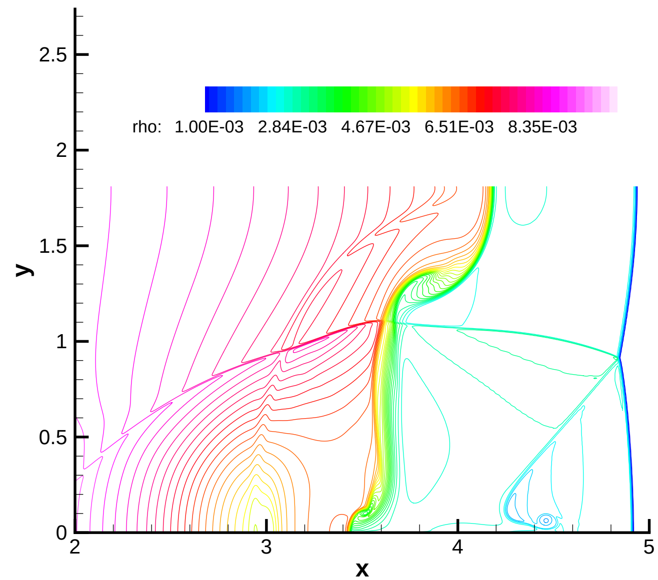
# Richtmyer-Meshkov Instability (Cont.)

$(M = 4.46, a = 10\ mm, \delta = 6\ mm, t = 50\ \mu s)$ **: Density**

**Spectral**




**WENO**

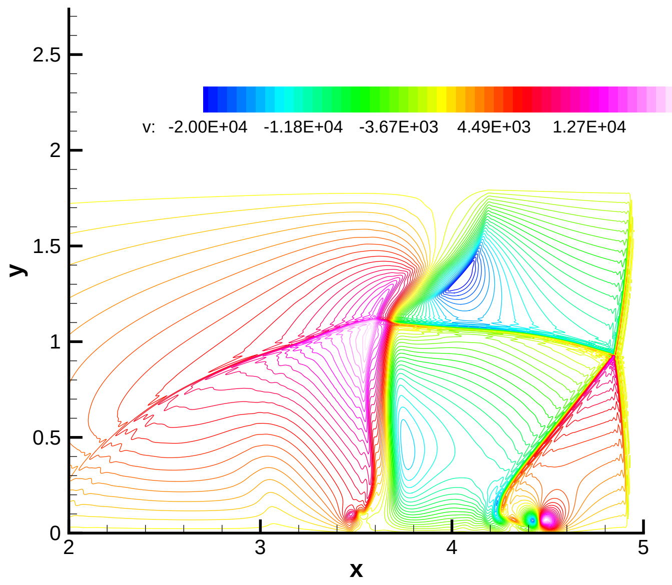


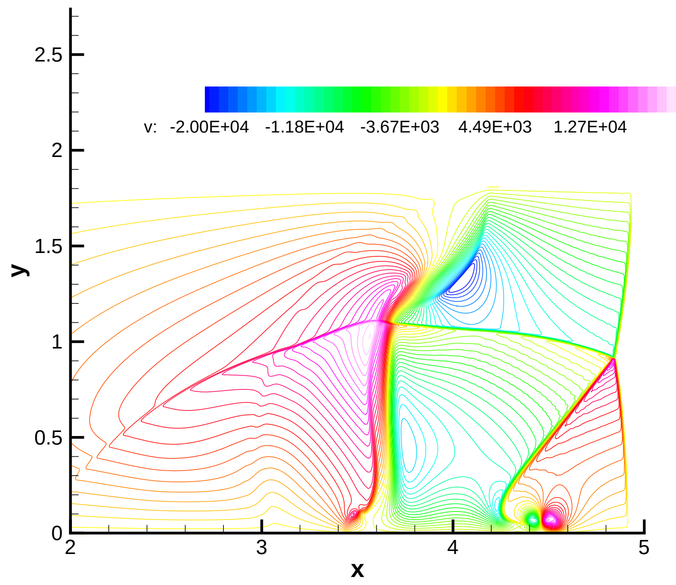

- Grid size for the Spectral and WENO schemes are 1024x512.

## Richtmyer-Meshkov Instability (Cont.)

$(M = 4.46, a = 10\ mm, \delta = 6\ mm, t = 50\ \mu s)$ **: V-Velocity**

**Spectral** **WENO**



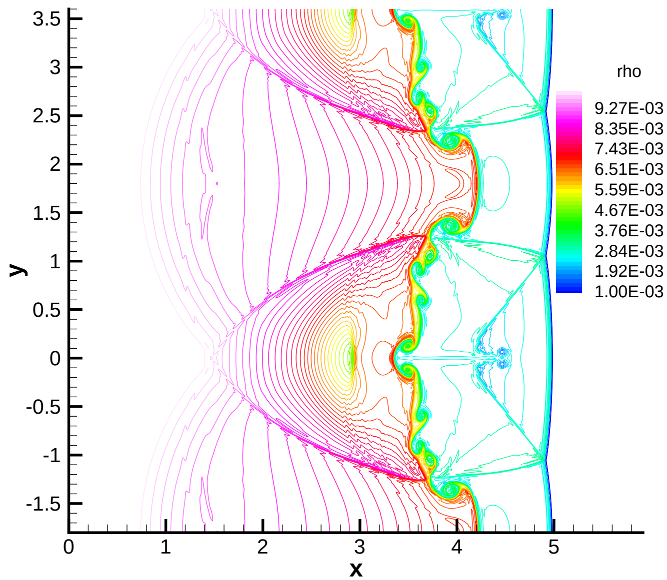




- Grid size for the Spectral and WENO schemes are 1024x512.

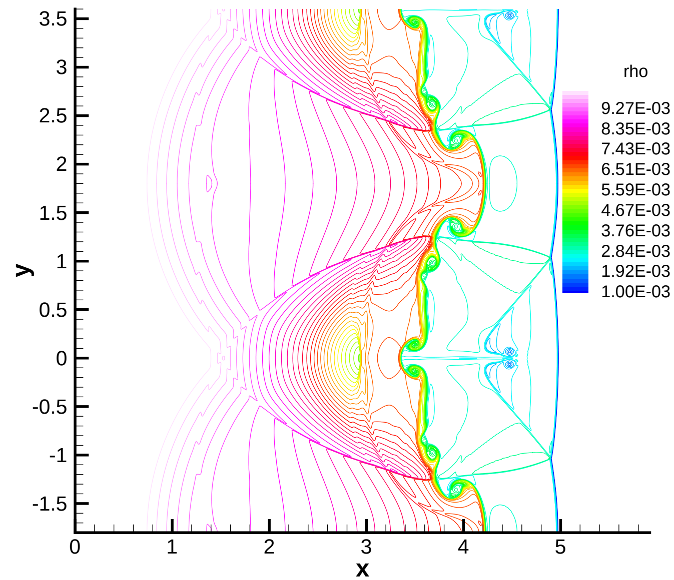
# Richtmyer-Meshkov Instability (Cont.)

$(M = 4.46, a = 10\ mm, \delta = 2\ mm, t = 50\ \mu s)$ **: Density**

**Spectral**




**WENO**

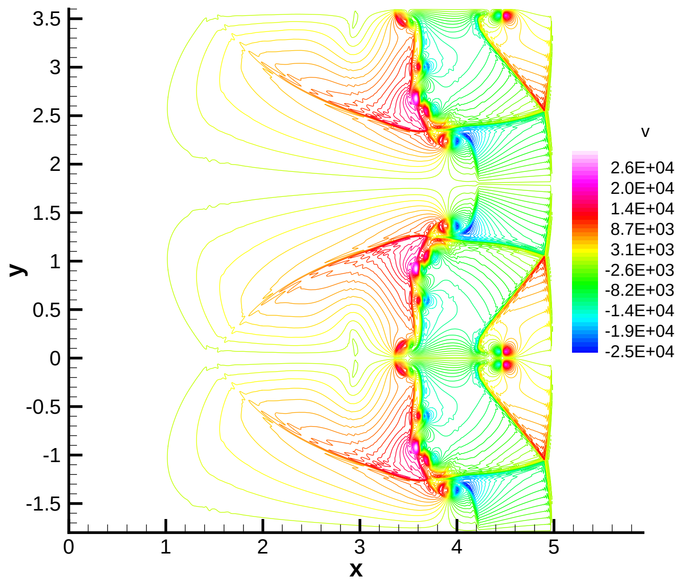



- Grid size for the Spectral and WENO schemes are 1024x256 and 1024x512 respectively.

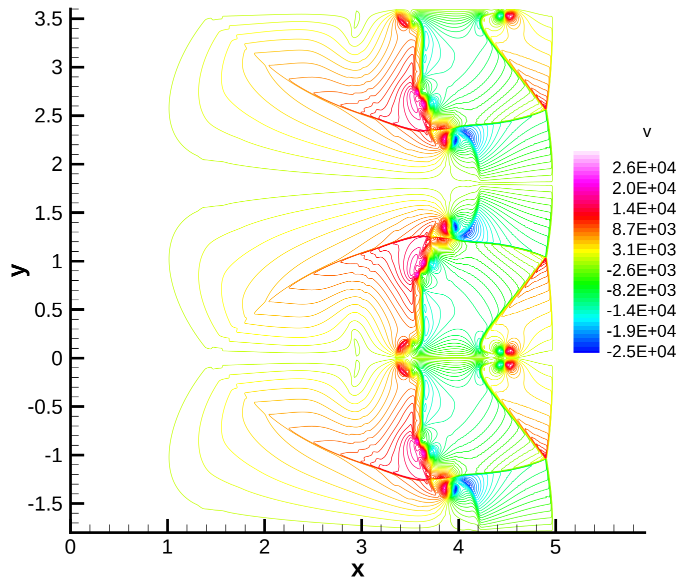
## Richtmyer-Meshkov Instability (Cont.)

$(M = 4.46, a = 10\ mm, \delta = 2\ mm, t = 50\ \mu s)$ **: V-Velocity**

**Spectral**




**WENO**

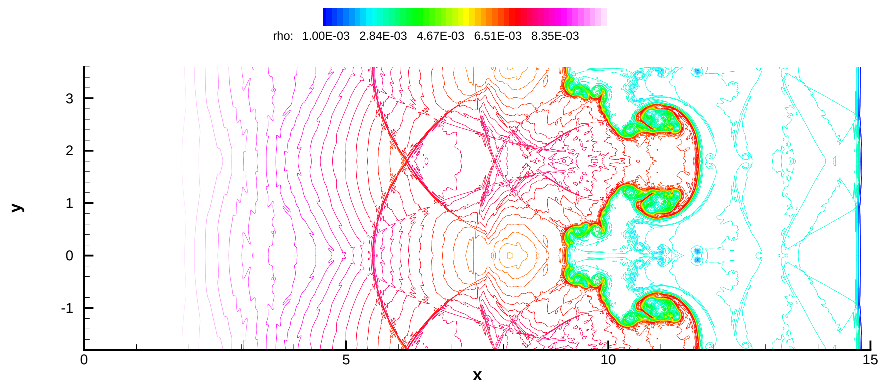



- Grid size for the Spectral and WENO schemes are 1024x256 and 1024x512 respectively.

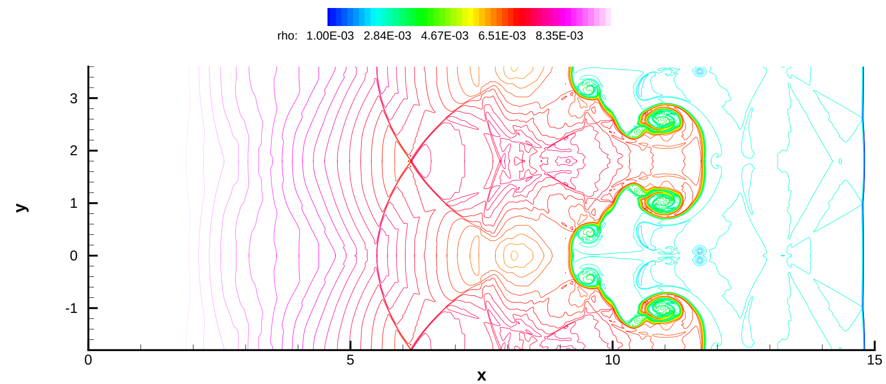
# Richtmyer-Meshkov Instability (Cont.)

**Long time Case ($M = 4.46, a = 10\ mm, \delta = 2\ mm, t = 124\ \mu s$) : Density**

**Spectral**



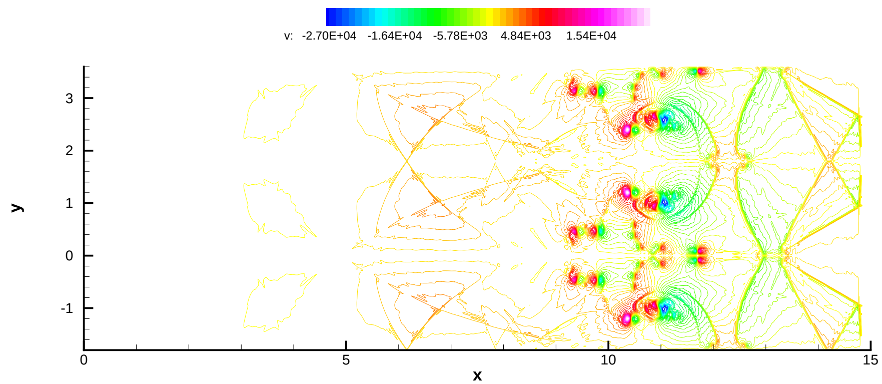

**WENO**



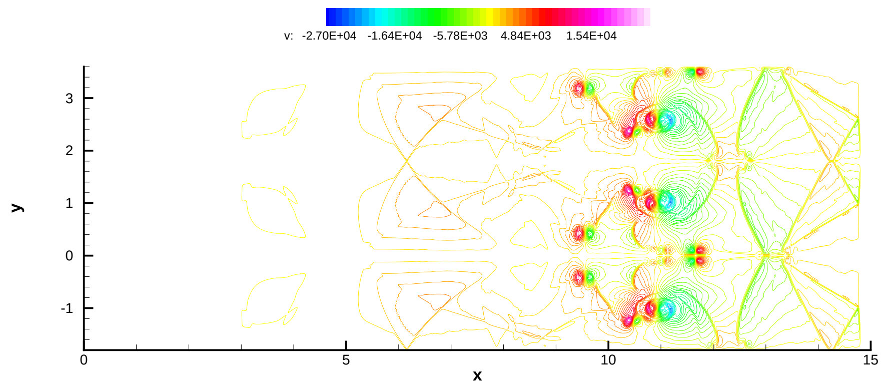
# Richtmyer-Meshkov Instability (Cont.)

**Long time Case** $(M = 4.46, a = 10\ mm, \delta = 2\ mm, t = 124\ \mu s)$ **: V-Velocity**

**Spectral**



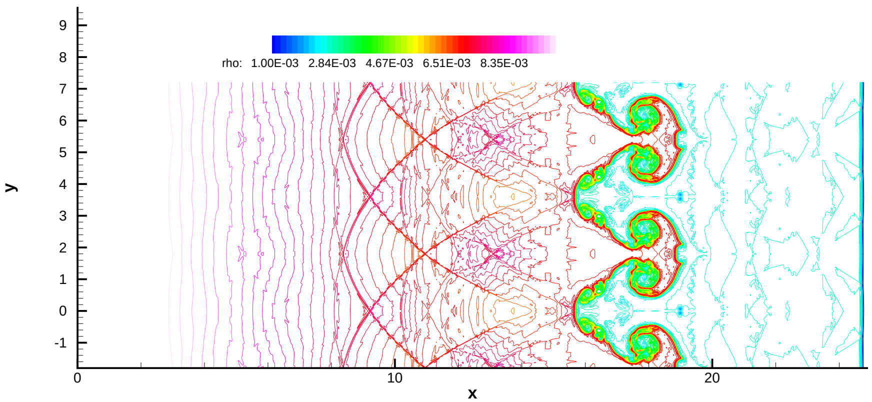

**WENO**



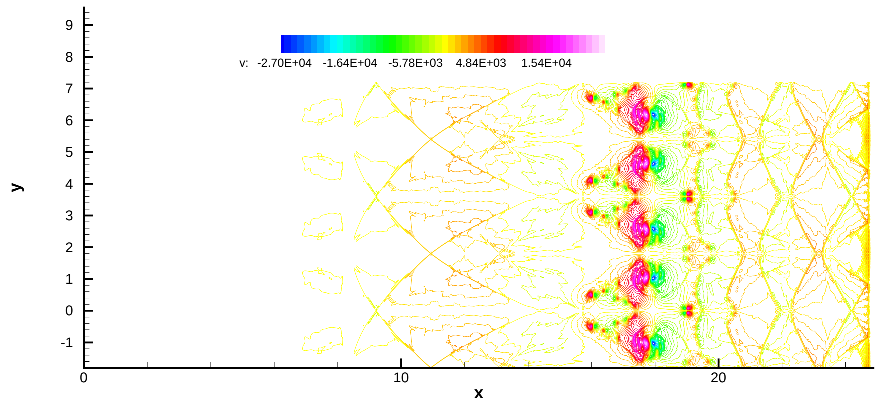
# Richtmyer-Meshkov Instability (Cont.)

**Large Domain Case ($M = 4.46, a = 10\ mm, \delta = 2\ mm, t = 237\ \mu s$) : Spectral scheme**

**Density**



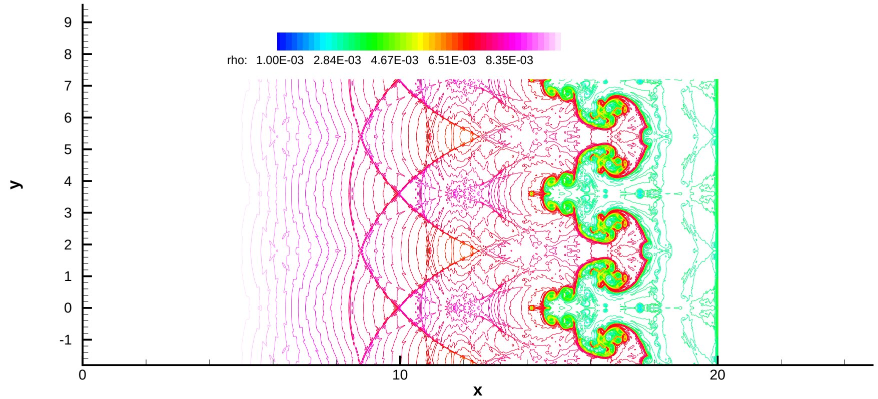

**V-Velocity**



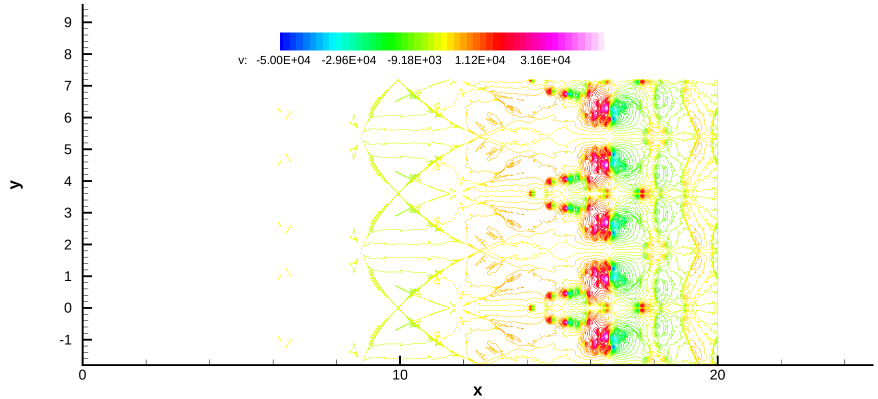
## Richtmyer-Meshkov Instability (Cont.)

**High Mach Number ($M = 8, a = 10\ mm, \delta = 2\ mm, t = 200\ \mu s$) : Spectral scheme**

**Density**

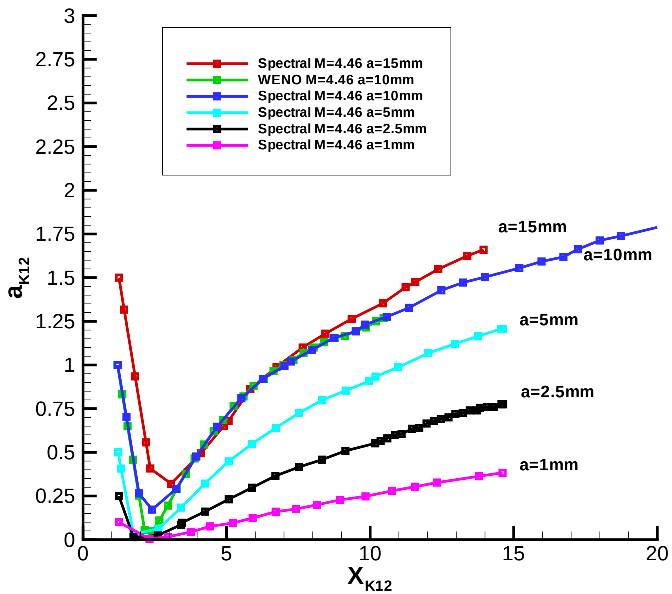


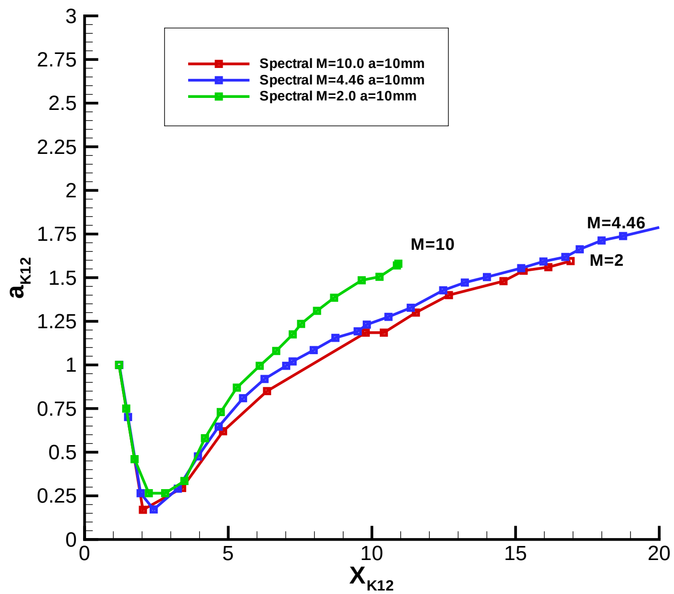

**V-Velocity**



# Richtmyer-Meshkov Instability (Cont.)

## Amplitudes vs. Displacement




- Consistent with the experimental observations by Aleshin et al.
  1. Linear growth of $a$ within "soft regular" regime;
  2. Decrease growth of $a$ within the irregular regime.

# Richtmyer-Meshkov Instability : Conclusion

**Summary :**

- Good agreement of the global large and medium features between the Spectral scheme and the WENO scheme.
- Some discrepancy of the fine scale vortical structures along the gaseous interface as expected for simulations of this sensitive nature to small perturbation (physically and/or numerically).
- Good agreement with the experimental data (Aleshin et al.)