An Efficient and High Resolution Solver for the Two-Dimensional Numerical Simulation of the Richtmyer-Meshkov Instability

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Key features of the Solver

The solver based on the model developed byWang *et al.* (2001), which uses a thermodynamically consistent fully conservative approach for the treatment of contact discontinuities based upon the concept of Total Enthalpy Conservation of the Mixture (ThCM), is implemented to study the Richtmyer-Meshkov instability. This method utilizes a high resolution Godunov-type scheme based upon a fast exact Riemann solver and the Piece-wise Spline Method (PSM) for data reconstruction of primitive variables at cell interfaces with fourth order accuracy.

ThCM Model

In the two-dimensional simulation of the Richtmyer-Meshkov instability, the hyperbolic conservation laws with the ThCM model are given by:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = 0$$

where **U** are the conservative variables and **F** and **G** are the conservative fluxes in the *x* and *y* directions respectively.

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\chi \\ \rho u \\ \rho v \\ H-p \end{pmatrix}; \quad \mathbf{F} = \begin{pmatrix} \rho u \\ \rho u\chi \\ \rho u^2+p \\ \rho uv \\ uH \end{pmatrix}; \quad \mathbf{G} = \begin{pmatrix} \rho v \\ \rho v\chi \\ \rho uv \\ \rho v^2+p \\ vH \end{pmatrix}$$

where *H*, the total enthalpy per unit volume, is given by:

$$H = \chi p + \frac{1}{2}\rho\left(u^2+v^2\right)$$

In the model, the pressure is calculated following the current value of the conservative variable term (*H-p*) as follows:

$$p = \frac{(H-p) - \frac{1}{2}\rho\left(u^2+v^2\right)}{\chi - 1}$$

Where χ is introduced in the ThCM model to simplify the expression of the governing equations and is defined as:

$$\chi = \frac{\gamma}{\gamma-1} = \alpha_1\chi_1 + \alpha_2\chi_2$$

A splitting scheme is employed for the system in two spatial dimensions defined as follows:

$$(A)\quad for \quad x-sweep: \quad \begin{pmatrix} \rho \\ \rho u \\ \rho\chi \\ H-p \\ \rho v \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2+p \\ \rho u\chi \\ uH \\ \rho uv \end{pmatrix}_x = 0; \qquad (B)\quad for \quad y-sweep: \quad \begin{pmatrix} \rho \\ \rho v \\ \rho\chi \\ H-p \\ \rho u \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2+p \\ \rho v\chi \\ vH \\ \rho uv \end{pmatrix}_y = 0.$$

The tangential velocity component, *v(u)* in the *x*-sweep(*y*-sweep), is passively advected with the normal velocity component, *u(v)*. A two-step process shown is used to accomplish the integration from time *n* to *n*+1 using a Godunov scheme:

$$\mathbf{U}_{i,j}^{n+1/2} = \mathbf{U}_{i,j}^{n} + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i-1/2,j}^{n} - \mathbf{F}_{i+1/2,j}^{n}\right); \quad \forall j \quad (a)$$

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^{n+1/2} + \frac{\Delta t}{\Delta y}\left(\mathbf{G}_{i,j-1/2}^{n+1/2} - \mathbf{G}_{i,j+1/2}^{n+1/2}\right); \quad \forall i. \quad (b)$$

The PSM and following slope limiters, similar to those developed by Ren *et al. (1996)*, are used for the data reconstruction of the conservative variables for the local Riemann problem solutions at cell interfaces.

$$\mathbf{U}_i^L = \mathbf{U}_i^n - \frac{\Delta x_i}{2}\mathbf{m}_i + \frac{(\Delta x_i)^2}{8}\mathbf{M}_i - \frac{(\Delta x_i)^3}{48}\mathbf{M}_i^{(3)} \quad (a)$$

$$\mathbf{U}_i^R = \mathbf{U}_i^n + \frac{\Delta x_i}{2}\mathbf{m}_i + \frac{(\Delta x_i)^2}{8}\mathbf{M}_i + \frac{(\Delta x_i)^3}{48}\mathbf{M}_i^{(3)} \quad (b)$$

$$\mathbf{U}_i^L = \mathbf{U}_i^n - \frac{1}{2}\mathrm{minmod}\left(\mathbf{m}_{i-1},\mathbf{m}_i,\mathbf{m}_{i+1}\right)\Delta x_i + \frac{1}{8}\mathrm{minmod}\left(\mathbf{M}_{i-1},\mathbf{M}_i,\mathbf{M}_{i+1}\right)(\Delta x_i)^2 - \frac{1}{48}\mathrm{minmod}\left(\mathbf{M}_{i-1}^{(3)},\mathbf{M}_i^{(3)},\mathbf{M}_{i+1}^{(3)}\right)(\Delta x_i)^3$$

$$\mathbf{U}_i^R = \mathbf{U}_i^n + \frac{1}{2}\mathrm{minmod}\left(\mathbf{m}_{i-1},\mathbf{m}_i,\mathbf{m}_{i+1}\right)\Delta x_i + \frac{1}{8}\mathrm{minmod}\left(\mathbf{M}_{i-1},\mathbf{M}_i,\mathbf{M}_{i+1}\right)(\Delta x_i)^2 + \frac{1}{48}\mathrm{minmod}\left(\mathbf{M}_{i-1}^{(3)},\mathbf{M}_i^{(3)},\mathbf{M}_{i+1}^{(3)}\right)(\Delta x_i)^3$$

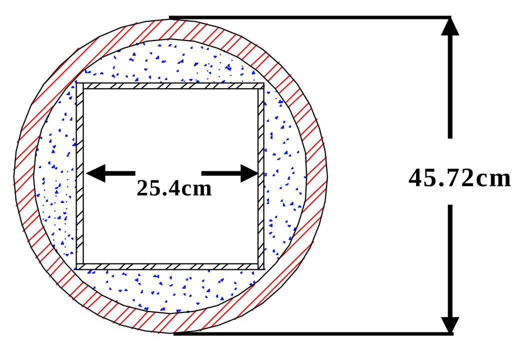
and

$$\mathrm{minmod}(g_{i-1},g_i,g_{i+1}) = \begin{cases} \min\left(|g_{i-1}|,|g_i|,|g_{i+1}|\right) sign(g_i) & if \ \min(g_{i-1},g_i,g_{i+1}) \times \max(g_{i-1},g_i,g_{i+1}) > 0 \\ 0 & else \end{cases}$$

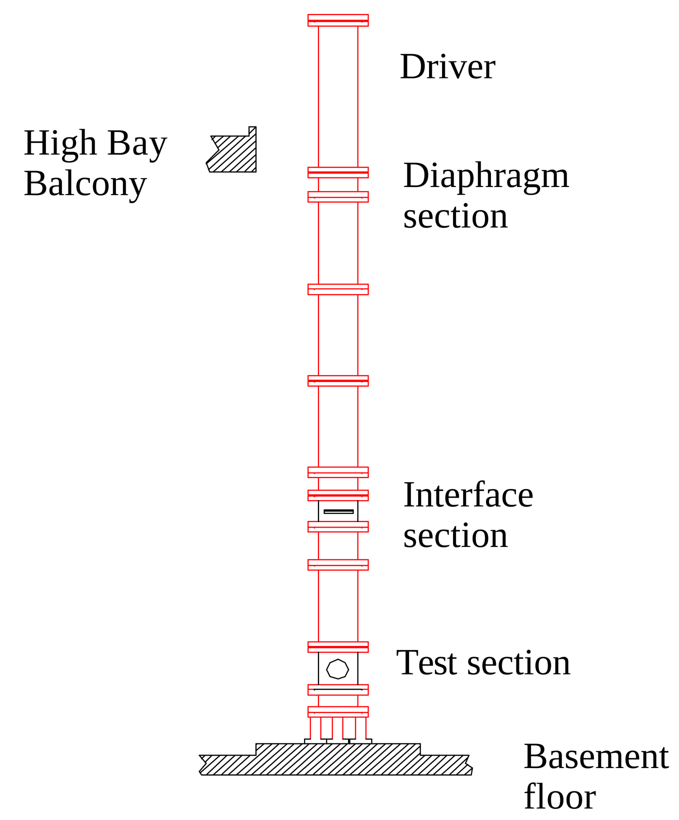
Simulations of the Wisconsin RM Shock Tube Experiments

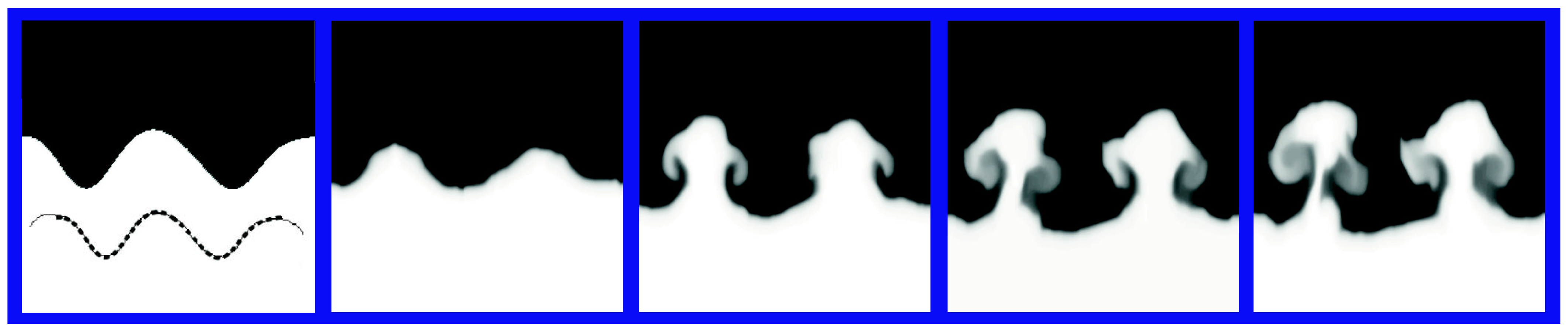
Total Length 9.2 m
Driven Length 6.8 m
Inner square cross section 0.25 m
Maximum Driver Pressure 100 atm

A series of two Richtmyer-Meshkov instability experiments were conducted in the University of Wisconsin's shock tube and have been simulated with the numerical model. In these experiments, the distance from the interface to the center of the test section was 0.457 m and an initial condition was created by the retraction of a sinusoidal copper plate which resulted in a Rayleigh-Taylor instability. The RT formed initial condition just prior (< 10 ms) before the shock interaction was recorded, and a sine series representation of the initial condition was used as the input initial condition for the simulation. The parameters of each of the experiments and the details of the calculations are presented below.



Driven cross section of shock tube



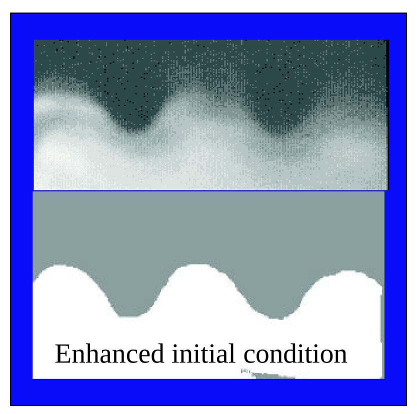


t=0 ms | *t*=0.17 ms | *t*=0.47 ms | *t*=0.72 ms | *t*=0.87 ms

Test 319	Shocked Ar	Unshocked Ar	Unshocked Air
Density ($\times 10^{-3}$ g/cm^3)	46.83	16.22	11.38
Pressure (bar)	9.68302	1	1
Mass velocity (km/sec)	0.5908	0	0
γ	1.667	1.667	1.4

Numerical Domain 2480x200 cells with a spatial resolution of 0.381mm
CPU time ~24 h on a 1.13 GHz Pentium 4

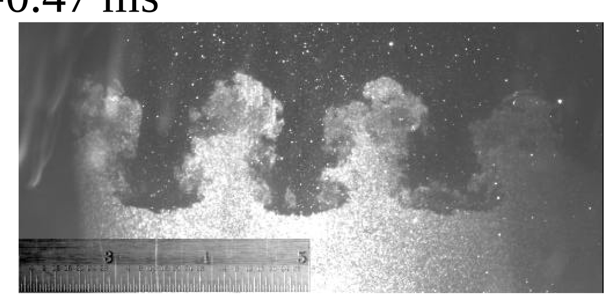
The growth rate from the simulation at the time of the experiment is 9.4 m/s.



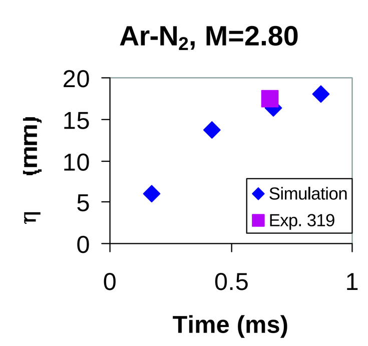
Initial condition

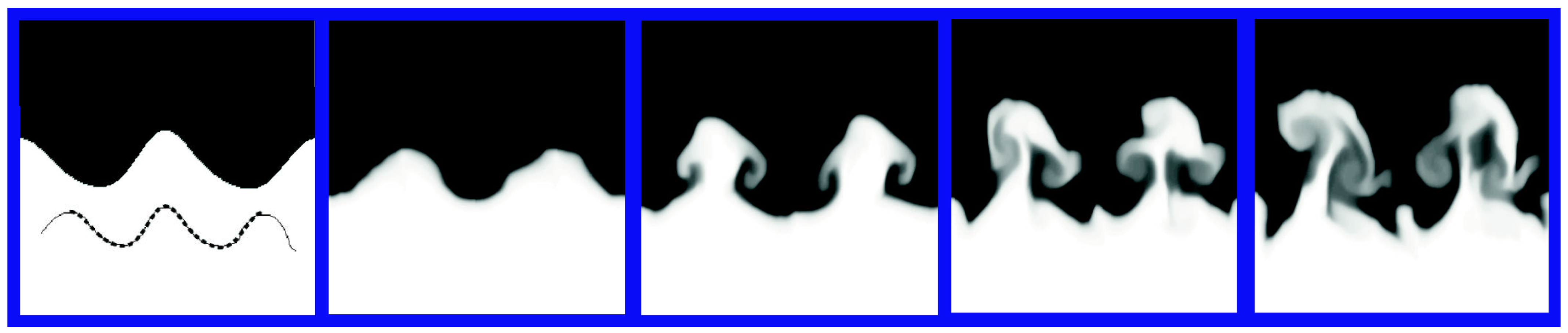
$$\eta_{IC}(x) = \sum_{i=1}^{5} a_i \sin\left(\frac{i\pi}{\lambda}x + \phi_i\right)$$

Coefficients of sine series	Expt.319 Ar/N_2 M=2.80
a_1 (mm)	0.811
a_2 (mm)	7.13
a_3 (mm)	0.736
a_4 (mm)	0.996
a_5 (mm)	0.198
ϕ_1	6.04
ϕ_2	1.98
ϕ_3	0.0359
ϕ_4	5.45
ϕ_5	5.64



Experimental image at t=0.70 ms after shock interaction. Measured amplitude η=17.52 mm



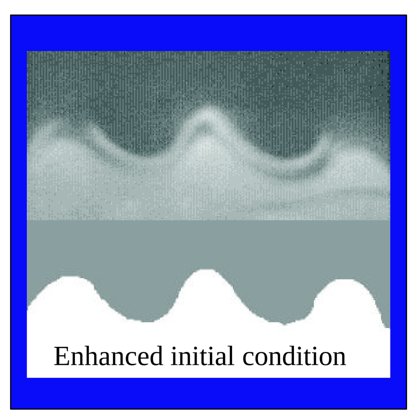


t=0 ms | *t*=0.17 ms | *t*=0.42 ms | *t*=0.67 ms | *t*=0.87 ms

Test 333	Shocked CO_2	Unshocked CO_2	Unshocked Air
Density ($\times 10^{-3}$ g/cm^3)	80.87	17.88	11.77
Pressure (bar)	10.72	1	1
Mass velocity (km/sec)	0.6504	0	0
γ	1.297	1.297	1.402

Numerical Domain 2480x200 cells with a spatial resolution of 0.381mm
CPU time ~24 h on a 1.13 GHz Pentium 4

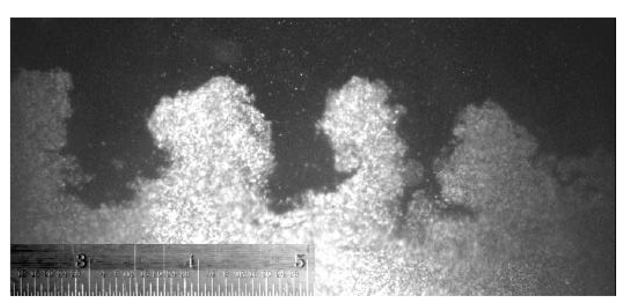
The growth rate from the simulation at the time of the experiment is 16.6 m/s.



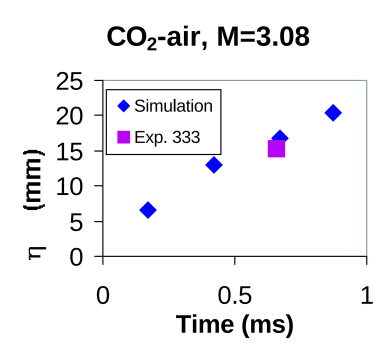
Initial condition

$$\eta_{IC}(x) = \sum_{i=1}^{5} a_i \sin\left(\frac{i\pi}{\lambda}x + \phi_i\right)$$

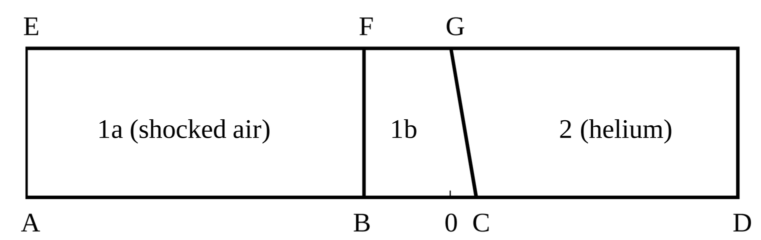
Coefficients of sine series	Expt.333 CO_2/Air M=3.08
a_1 (mm)	0.747
a_2 (mm)	6.80
a_3 (mm)	0.365
a_4 (mm)	0.916
a_5 (mm)	0.118
ϕ_1	5.28
ϕ_2	1.50
ϕ_3	5.41
ϕ_4	2.03
ϕ_5	5.30



Experimental image at t=0.66 ms after shock interaction. Measured amplitude η=15.29 mm



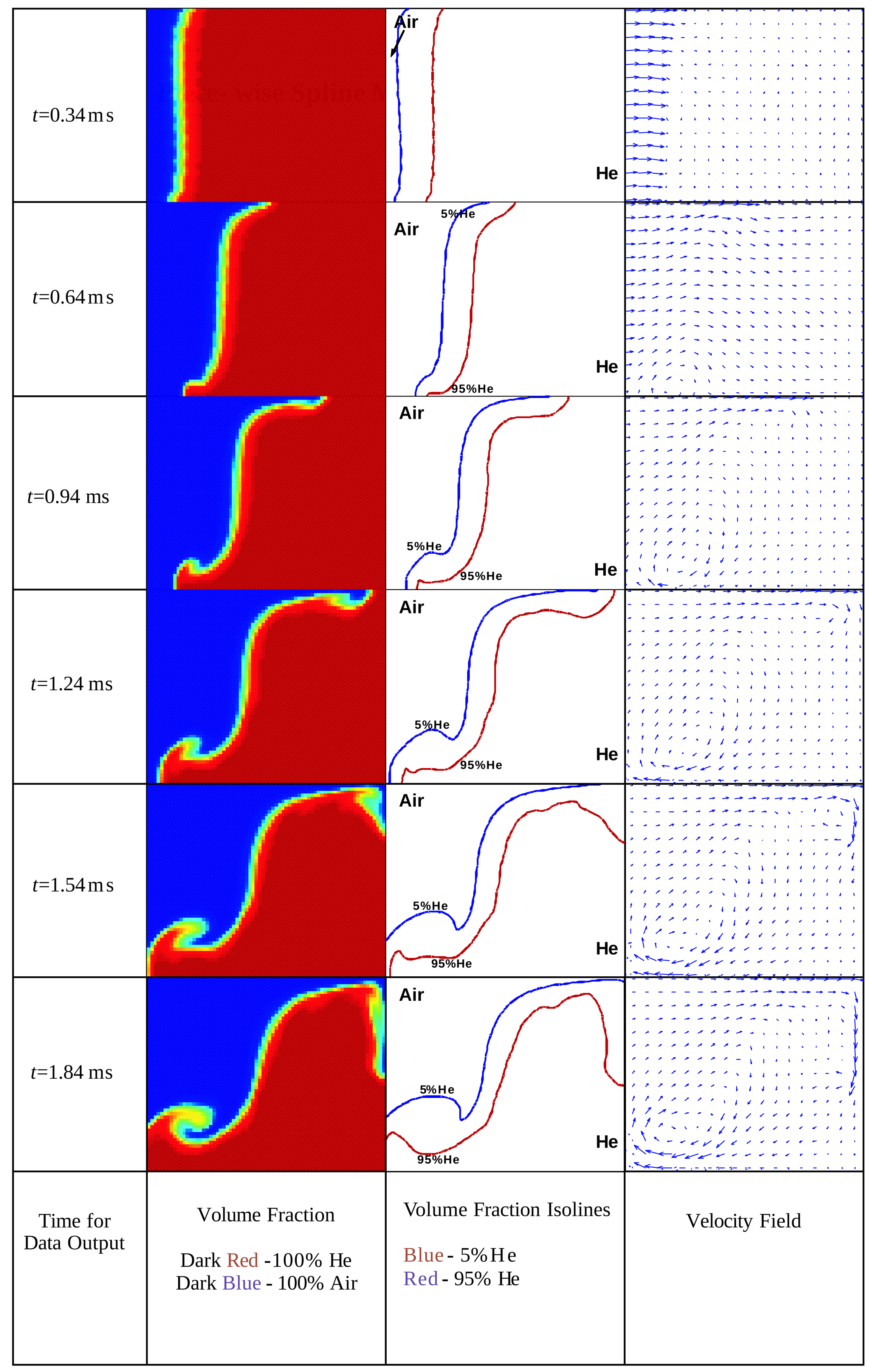
Simulation of Test Problem #1



Boundary condition
Inflow condition of air with parameters behind shock is set on the left boundary and other boundaries are rigid walls.

Mesh
The mesh is square for x>-2cm. Zone size of 0.2 cm and 60 zones per region width.

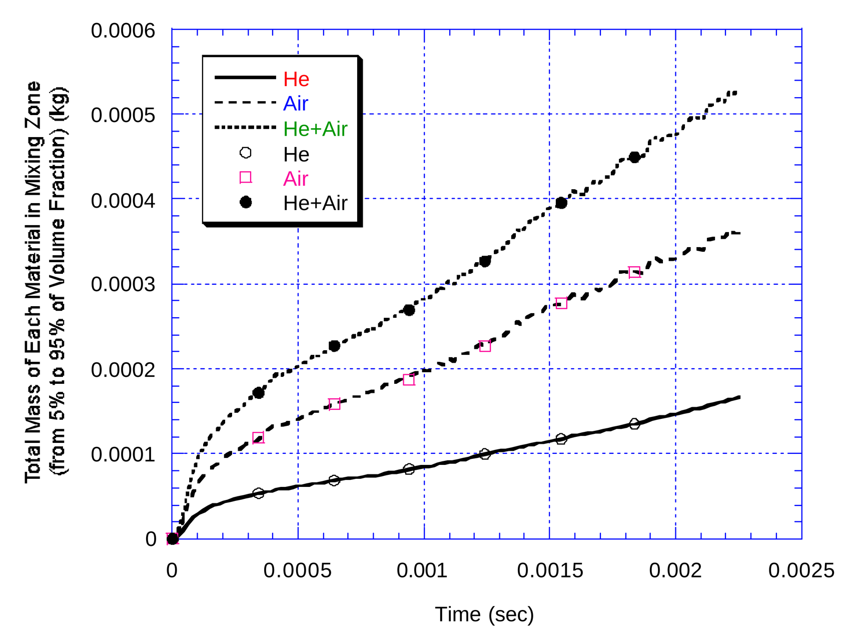
AB=38cm, BC=3.2cm, 0D=20cm, AE=12cm, FG=0.8cm. Initially the average position of the interface is x=0 (point 0). Regions 1a,b contain Air (1a - shocked air, 1b - unshocked air), region 2 contain He. The incident shock moves from left to right, i.e. From Air to He. The full width of the tube is 12 cm and the full perturbation amplitude is 2.4 cm.



Initial conditions

	Shocked Air	Unshocked Air	Unshocked He
Density ($\times 10^{-3}$ g/cm^3)	18.2641	12.05	1.67
Pressure (bar)	1.804997	1	1
Mass velocity (km/sec)	0.15076	0	0
γ	1.4	1.4	1.63

Results
The left most series of images shows the volume fractions of the two gas species as a function of time. The center set of images are 5% and 95% volume fraction isolines. The third column shows the local velocity vectors
.
The graph below shows the total mass of each material in the mixing zone (from 5% to 95% of the volume fraction). The points on the graph correspond to the times in the images on the left.



Wang, S.P., Anderson, M.H., Oakley, J.G., Corradini, M.L., Bonazza, R., A thermodynamically consistent and fully conservative treatment of contact discontinuities for compressible multicomponent flows submitted to *J. Comput. Phys.* (2001)

Ren, Y.X., Liu, Q.S.,Wang, S. P., Shen, M. Y., A high order accurate, non-oscillating finite volume scheme using spline interpolation for hyperbolic conservation laws, *ActaAerodynamica Sinica* **14**, 281 (1996)