Semiempirical model of turbulent magnetic field diffusion to driven plasma

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Paper to be presented at the 8th IWPCTM, Pasadena, 2001.

There are quite many astrophysical and geophysical problems [1,2] as well as CTF related problems [3], in which a significant role is played by plasma-magnetic field interface instability. Among the numerous types of the instability, the magnetohydrodynamic instabilities were studied first. A most well-known example of this instability type is a so-called "chute" instability arising at the plasma-magnetic field interface during the interface acceleration.

This instability type is studied in a large number of papers, (see ref. [3]). The papers, as a rule, consider the linear stage of its evolution. The studies of the later, nonlinear stage became possible only recently thanks to availability of numerical methods. The approaches can be exemplified by refs. [4,5,8]. The first of them treats collisionless plasma and uses a "hybrid" model, the others consider plasma as "collisional" and perform the computation in the MHD approximation.

It was possible to obtain many useful results for the RT instability evolution at the plasma-magnetic field interface using this kind of approaches, however, all examples known to us of using these approaches are limited to problems, where unperturbed flow is one-dimensional. (A typical example is plasma cylinder expansion.) This limitation relates to the fact that such approaches prove quite complex, cumbersome, and do not allow us to follow the computed flow behavior in quite small scales.

A similar difficulty is encountered in unstable flow computations in hydrodynamics. A method to avoid this is using "semiempirical" turbulent mixing (TM) models. The model for "gravitational" TM problems was formulated for the first time in ref. [6], later the "semiempirical" models were developed quite extensively, became more complex, and found wide use for computation of a different kinds of turbulent flows. The models can be exemplified with those of refs. [7,12,13].

The studies of the RT instabilities in hydrodynamics and MHD flows revealed quite a close analogy between them at the linear stage [3]. This fact allows us to expect that this analogy may be also valid for a later, "turbulent" stage of the problem.

This paper discusses a semiempirical model for computing characteristics of a transitional layer at the accelerated plasma cloud – magnetic field interface. The model was developed on the basis of the hydrodynamic models [6,7] with using the above-mentioned analogy. Results of some 1D and 2D computations by the model are presented.

1. Derivation of governing equations

Assume that the flow can be described by MHD equations [9].

Thus, we have:

$$\frac{\partial \rho}{\partial t} + \text{div}\rho\vec{u} = 0 \tag{1.1}$$

$$\frac{\partial(\rho\vec{u})}{\partial t} + \text{Div}(\vec{S}) = \vec{f} \tag{1.2}$$

$$\frac{\partial Q}{\partial t} + \text{div}\left[\rho\vec{u}\left(\frac{u^2}{2} + \omega\right) - (\vec{u}\cdot\vec{\sigma}) + \frac{1}{4\pi}\left[\vec{H}\vec{E}\right]\right] = 0 \tag{1.3}$$

$$\vec{f} = \frac{1}{4\pi}\left[\text{rot}\vec{H}\cdot\vec{H}\right] \tag{1.4}$$

where $S_{ik} = P\delta_{ik} + \rho u_i u_k - \sigma_{ik}$ \tag{1.5}

$$Q = \frac{\rho u^2}{2} + \rho E + \frac{H^2}{8\pi} \tag{1.6}$$

$$\vec{E} = \frac{1}{c}\left[\vec{u}\vec{H}\right] - \frac{c}{4\pi\Sigma}\text{rot}\vec{H} \tag{1.7}$$

$$\omega = E + \frac{P}{\rho} \tag{1.8}$$

$\vec{\sigma}$ is viscous stress tensor, Σ - conductivity, other notations are conventional.

This system is complemented with equations for magnetic field:

$$\frac{\partial\vec{H}}{\partial t} + (\vec{u}\nabla)\vec{H} = (\vec{H}\nabla)\vec{u} + \nu_m\Delta H \tag{1.9}$$

$$\text{div}\vec{H} = 0 \tag{1.10}$$

where $\nu_m = \dfrac{C^2}{4\pi\sigma}$ is "magnetic" viscosity factor, as well as with the equation for different concentrations α_i:

$$\frac{\partial \rho_i}{\partial t} + \mathrm{div}\rho_i \vec{u} = \mathrm{div}(\rho i\ \nabla\alpha_i) \qquad (1.11)$$

where $\rho_i = \rho\alpha_i$, D − coefficient of molecular diffusion, and equation of state:

$$P = P(\rho, E, \alpha_i) \qquad (1.12)$$

As usual, divide the quantities to be calculated into low- and high-frequency addends, then make appropriate averaging [10,11], and use the commonly accepted expressions for the addends containing the third or higher order moments to obtain the following from set 1.1 through 1.12 for averaged values:

For density: Eq. 1.1

For velocity: Eq. 1.2

For concentrations : Eq. 1.11

For "magnetic" force: Eq. 1.4.

The equations for velocity and density therewith involve the "turbulent" tensor of viscous pressures, σ_t, and diffusion coefficient, D_t, instead of relevant "molecular" values.

For internal energy:

$$\frac{\partial}{\partial t}(\rho \cdot E) + div(\rho E \vec{u}) = c_\alpha div(\rho D \nabla(\alpha E)) - Pdiv\vec{u} + \rho\varepsilon + \frac{(\vec{j})^2}{\Sigma_T} \qquad 1.13$$

$$\vec{j} = \frac{c}{4\pi} rot\vec{H} \qquad 1.14$$

Here ε is dissipation rate of turbulent energy k, Σ_T is effective conductivity.

The following relations are used for functions σ_t and D_t [7]:

$$D_t = C_D \frac{k^2}{\varepsilon} \qquad 1.15$$

$$\sigma_t^{ij} = \rho D_t \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_r}{\partial x_r} \right] - \frac{2}{3}\rho k \delta_{ij} \qquad 1.16$$

Also, note that relations 1.11 to 1.16 were derived assuming Reynolds numbers Re and $Re_{mag} \to \infty$ and addends, such as "turbulent pressure" P_T and "magnetic turbulent pressure, were neglected.

A number of additional assumptions were used in this paper to calculate averaged magnetic field:

- cylindrical symmetry of the problem was assumed,
- the geometry of the initial magnetic field was assumed "poloidal",
- "magnetic viscosity" can be neglected.

Under these assumptions, the magnetic field components H_ρ and H_z (where z is axis of symmetry) are related with A_ϕ, azimuthal potential vector component, as

$$H_\rho = -\frac{\partial A_\phi}{\partial z} \qquad\qquad 1.17$$

$$H_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(A_\phi \rho) \qquad\qquad 1.18$$

$$A_\phi \rho = m \qquad\qquad 1.19$$

where

$$\frac{\partial m}{\partial t} + \vec{u}\nabla m = \text{div}\vec{R}_m \qquad\qquad 1.20$$

$$\vec{R}_m = -\langle \vec{u}'m' \rangle = \tilde{D}\nabla m \qquad\qquad 1.21$$

Assume that $\tilde{D} = C_m D$, where C_m is an empirical constant.

When considering the equations for turbulent values k and ε, turbulence anisotropization by magnetic field must be included. To do this, a simplest method is using the equation system for $k_{ij} = \langle u_i' u_j' \rangle$ instead of a single equation for k [7]. By analogy with this equation [7], the following can be written for diagonal components k_j:

$$\frac{\partial}{\partial t}(\rho k_j) = \rho(G_j - \varepsilon_j) + C_k div\rho D_j \nabla k_j + \Phi_j - \text{Re}l(\rho k_j) - div\rho k_j \vec{u} - \frac{2}{3}\rho k_j div\vec{u} \quad 1.22$$

$$G_j = G_j^{(1)} + G_j^{(2)} \qquad\qquad 1.23$$

$$G_j^{(1)} = \sigma_{jl}\frac{\partial u_j}{\partial x_l} \text{ - shear generation} \qquad\qquad 1.24$$

$$G_j^{(2)} = \frac{w_j \gamma_j}{\rho} \text{ - gravitational generation} \qquad\qquad 1.25$$

$$\vec{\gamma} = \nabla(P + P_m) - \frac{1}{4\pi}(\vec{H}\nabla\vec{H}) \qquad 1.26$$

$$\vec{w} = \frac{\nabla P}{\gamma P} - \frac{\nabla \rho}{\rho} \quad \vec{w} \text{ - turbulent flow} \qquad 1.27$$

$$\text{Re } l(\rho k_j) = \frac{\varepsilon}{k}\rho\left(k_j - \frac{k}{3}\right) \qquad 1.28$$

Φ_j - exchange term (between velocity and magnetic field fluctuation)

$$\Phi_j = \frac{1}{4\pi}\cdot 2\cdot\left[H_l < u_j\frac{\partial h_j}{\partial x_l}> + \frac{\partial H_j}{\partial x_l}r_{jl} - \left\langle u_j\frac{\partial}{\partial x_j}P'_m\right\rangle\right] \qquad 1.29$$

here $r_{jl} = \langle u_j h_l \rangle$, $P'_m = \left(H_k + \frac{h_k}{2}\right)h_k$, h_k - fluctuating part of magnetic field, ε_j -

dissipation of the k_j.

To "close" system 1.21-1.29, Φ_j and ε_j should be expressed in terms of the previously introduced functions.

The last addend in 1.29 can be written (by analogy with Rotta's hypothesis [10]) as a sum of the diffusion and relaxation terms for tensor components $\pi_{ik} = \langle h_i h_k \rangle$. If then the first addend in 1.29 is neglected, it is possible to express function Φ_j in terms of r_{jl} and π_{ik}.

The equation system for these functions can be derived from 1.2 and 1.9. By solving this system, they can be expressed in terms of k_{ij} and average velocities and fields.

The equation for ε_j can be written, following [11], as

$$\varepsilon_j = \frac{k_j^{\frac{3}{2}}}{l_j} \qquad 1.31$$

where l_j is the mixing path length that depends on the orientation of unit vector $\vec{e}_j$ relative to the magnetic field. When $\vec{e}_j \parallel \vec{H}$ the $l_j \to \infty$ if $\vec{H} \to \infty$ [11].It is suggested that the methods for determination of Φ_j and ε_j should be discussed separately.

2. Setting up the computations and discussion

The computations discussed below assumed that the magnetic field was not very strong and its effect on the anisotropy and turbulence dissipation could be neglected. This allowed us to neglect addend Φ_j in 1.22, i.e. to use one equation for k and one equation for ε presented in [7].

In so doing the magnetic field action on plasma is expressed in terms of force $\vec{f}$ in equation 1.2 and by relevant addends $\bar{\gamma}$ (1.26) in "generation" term G_2 for turbulent energy.

The plasma action on magnetic field leads to its displacement, however turbulence causes field diffusion into the plasma, with the diffusion coefficient itself depending on the field.

Two problems of plasma cloud expansion to surrounding, "background", low-density plasma ("vacuum") with magnetic field were calculated.

Problem 1. One-dimensional problem of cylindrical plasma cloud expansion in magnetic field.

It is agreed that at t=0 there is a cylindrical symmetric plasma cloud of radius r_0, energy E, and mass M (per unit length) surrounded by cold "background" plasma of density ρ_0 and by magnetic field oriented along axis of symmetry z, i.e. $\vec{H} = (0,0,H_1)$.

It is convenient to consider this problem in dimensionless variables. Introduce scaling: $u_1 = \left(\dfrac{E}{M}\right)^{\frac{1}{2}}$ for velocity, $r_1 = \left(\dfrac{E}{P_1}\right)^{\frac{1}{2}}$ for distance, $\dfrac{r_1}{u_1}$ for time, $\dfrac{M}{r_1^2}$ for density, H_1 for magnetic field (here P_1 is unperturbed magnetic field pressure).

In this scaling the problem is characterized with two dimensionless parameters, r_0' and $\rho_0' = \dfrac{\rho_0}{\rho_1} = \left(\dfrac{u_1}{C_A}\right)^2$, where $C_A^{\ 2} = \dfrac{H_1^{\ 2}}{4\pi\rho_0}$.

Next, restrict our consideration to $\rho_0' \to 0$, i.e. "sub-Alfven" plasma expansion.

The dimensionless initial data are summarized in Table 2.1

Table 2.1

	ρ	ε	u	H
$r < r_0$	$\dfrac{1}{\pi r_0'^2}$	1	0	0
$r > r_0$	ρ_0'	0	0	1

To calculate the turbulent mixing with the code of ref. [7], the data of initial turbulent energy profiles k and turbulent energy dissipation ε should be added to these initial data. It was assumed that at t=0 these functions are nonzero only in the layer of thickness $\delta r_0 \ll r_0$ near boundary r_0. In the computation, $C_m = 1$.

The computed data for $\rho_0' = \dfrac{10^{-2}}{\pi}$ and $r_0 = 0.1$ is plotted in Figs. 1.1-1.3.

The figures depict turbulent mixing zone (TMZ) boundaries with tine (here $R_1(t)$ is radius determined by level $H(R_1) = 0.05$, $R_2(t)$ is that by level $H(R_2) = 0.95$) (Fig. 1); magnetic field profiles at times $t = 0.5\tau$, and τ, where τ is period of one cylinder radius oscillation (Fig. 2), plasma density profiles at the same times (Fig. 3).

As seen in these figures, a noticeable magnetic field and plasma interpenetration ("turbulent diffusion") is observed. At $\rho_0' \ll 1$ and $r_0' \ll 1$ the result weakly depends on specific magnitudes of the parameters and on initial profiles k and ε.

Problem 2. Two-dimensional problem of spherical cloud expansion.

Assume that at t=0 there is a spherical plasma cloud of energy E, mass M, radius r_0 surrounded with "background" plasma of density ρ_0, thermal pressure P_0 and by magnetic field, whose strength tends to $\vec{H} = (0,0,H_0)$ with $r \to \infty$.

Like previously, introduce scaling $u_1^2 = \dfrac{E}{M}$ for velocity, $r_1 = \left(\dfrac{E}{P_1}\right)^{1/3}$ for distance,

$P_1 = \dfrac{\left(\frac{3}{2} H_0\right)^2}{8\pi} + P_0$ for pressure, $H_1 = H_0$ for magnetic field.

It was solved two spherical tasks №2.1 and 2.2. We assumed $H_0 = 0$, $P_0 \neq 0$ in the task 2.1 (so, this task become one-dimensional) and $H_0 \neq 0, P_0 = 0$ in the task 2.2.

The dimensionless initial data for the problem are summarized in Table 2.2.

Table 2.2

	ρ	ε	u	H
$r < r_0$	$\dfrac{3}{4\pi r_0^3}$	1	0	0
$r > r_0$	0,1	ε_0	0	H_0

Here ε_0=10, H_0=0 in the task 2.1 and ε_0=0, H_0=1 in task 2.2, r_0=0,5, $\gamma = 2$,

In contrast to Problem 1, here magnetic field in region $r > r_0$ is not constant and corresponds to the dipole magnetic field of radius r_0 in the external field of strength (0,0,1).

The initial data for the "turbulent" functions k and ε were given like in Problem 1: in a thin layer near interface r_0.

It was assumed that $C_m = 1$.

The results of the computation are plotted in the figures 4-9.

The first of them depicts the TMZ time dependence for 1D task 2.1. We can see TMZ width L_t is comparable with the cloud radious.

The fig. 5 depict the TMZ time dependence for 2D task 2.2 ($\theta = \dfrac{\pi}{2}$).

Comparing fig. 4 and 5 we can note although the plasma clouds dynamics are similar, but the magnetic field diminish value of L_t.

Figs. 6,7 plot the plasma density isolines at times t=0,4, 0,6, Figs. 8,9 depict the magnetic pressure isolines at the same times .

As seen, a noticeable field and plasma inter-diffusion is observed in this problem as well at the selected value of C_m, with this being significantly stronger at the $\theta = \dfrac{\pi}{4}$.

Conclusion

This paper discusses first our attempt to develop a semi-empirical model of turbulent mixing in inhomogeneous plasma with magnetic field.

To describe the MHD effects, the simplest form of the model is shown to require setting two additional constants determining the turbulent magnetic field diffusion factor and dissipative addends in the equation of energy.

To determine the constants, it is reasonable to involve experimental data on pinches and plasma liners.

It is being planned to make more accurate 2D computation and include turbulent energy anisotropy in it.

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Cylindrical cloud.
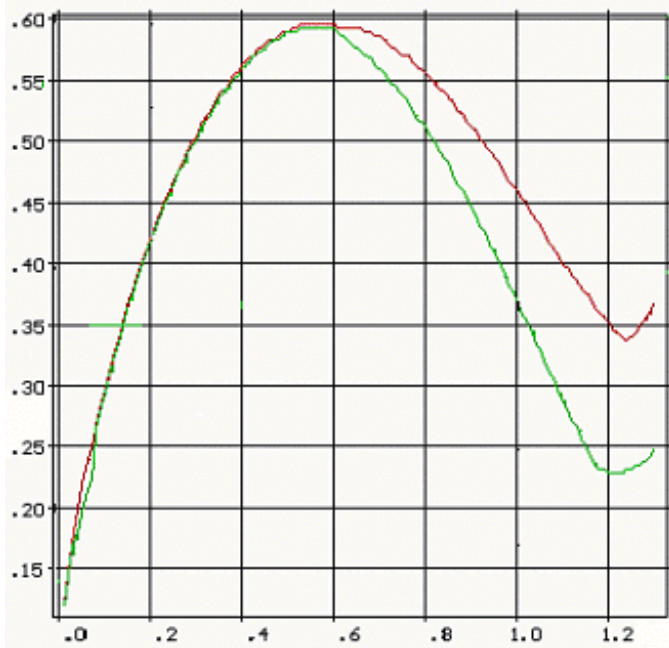
fig1. TMZ time dependence
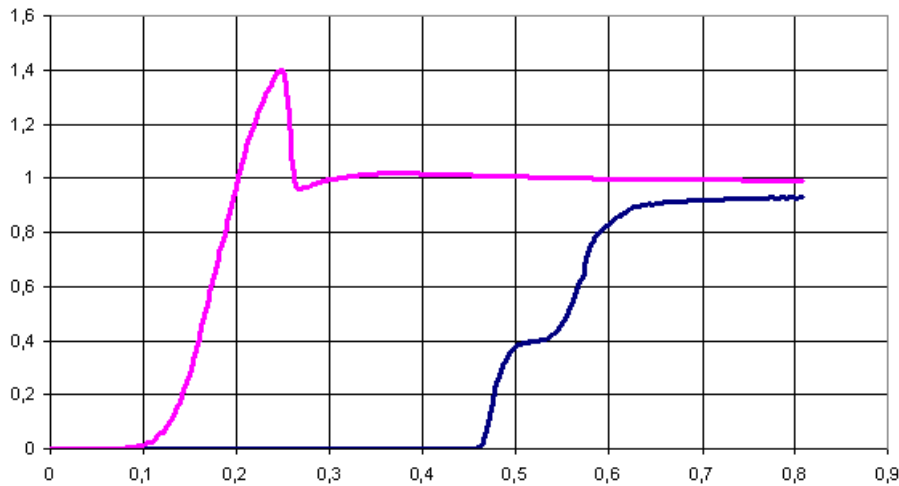(concentration levels 0.05 and 0.95), cylindr. task, P_m=1.0

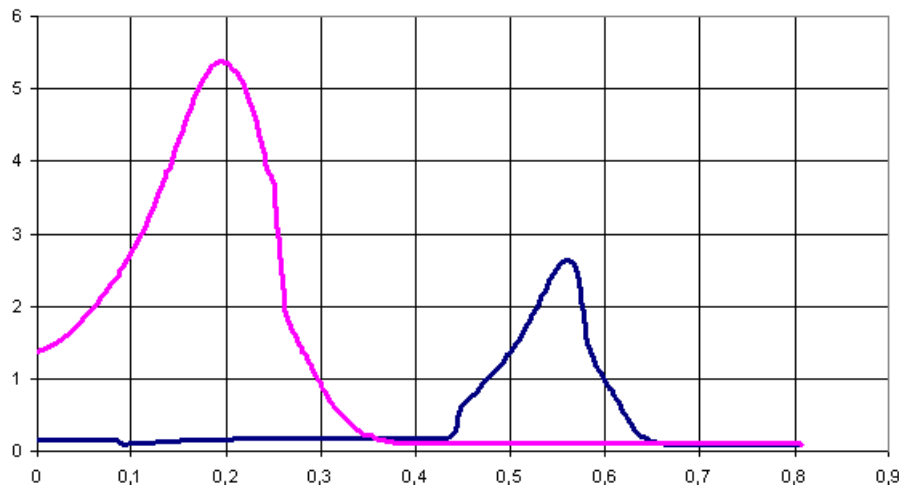fig2. Magnetic field profiles, cylindr. task, P_m=1.0, t=0.6, t=1.2

fig3. Plasma density profiles, cylindr. task, P_m=1.0, t=0.6, t=1.2
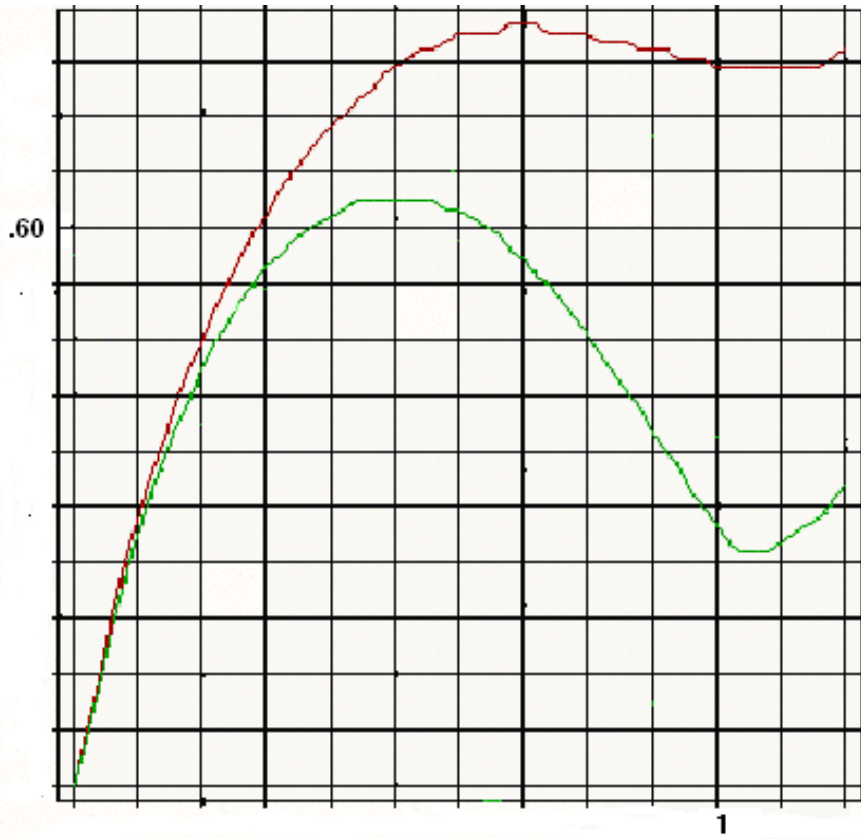
Spherical cloud.

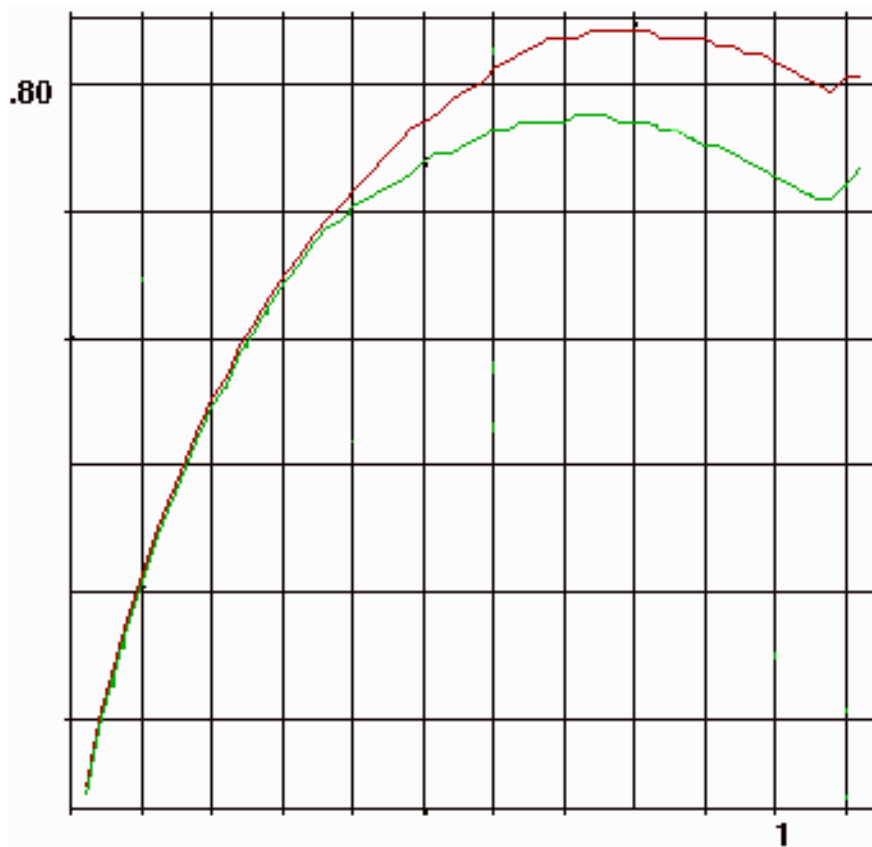fig4. TMZ time dependence, spher. task, P_{th}=0.1, P_m=0

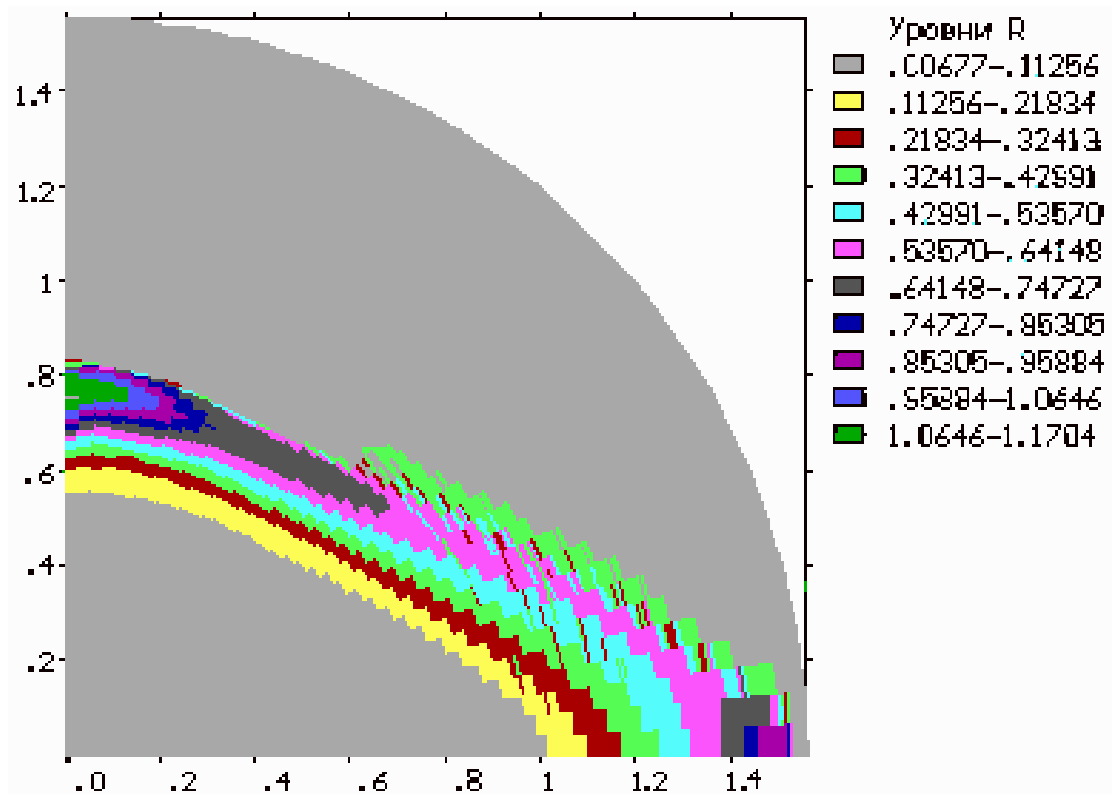fig5. TMZ time dependence, 2D task, P_m=0.44, Θ=$\pi/2$ (equator)

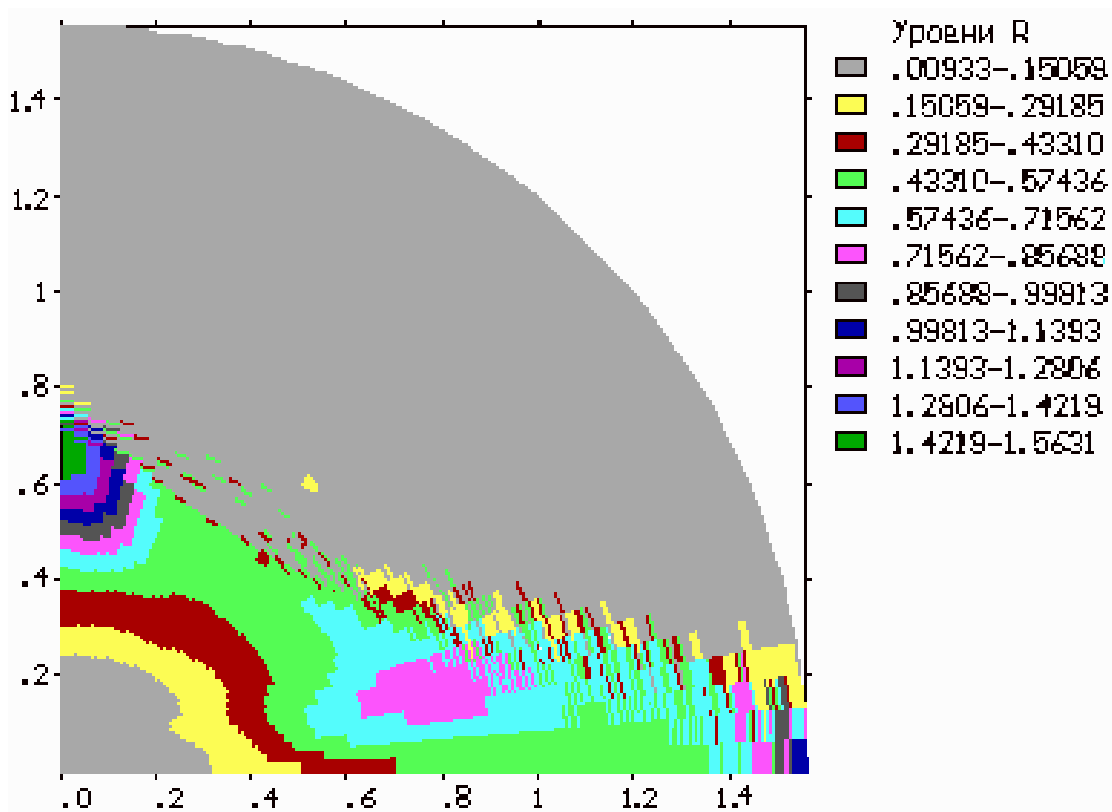fig6. Plot of the plasma density, 2D task, P_m=0.44, t = 0.8

fig7. Plot of the plasma density, 2D task, P_m=0.44, t = 1.2
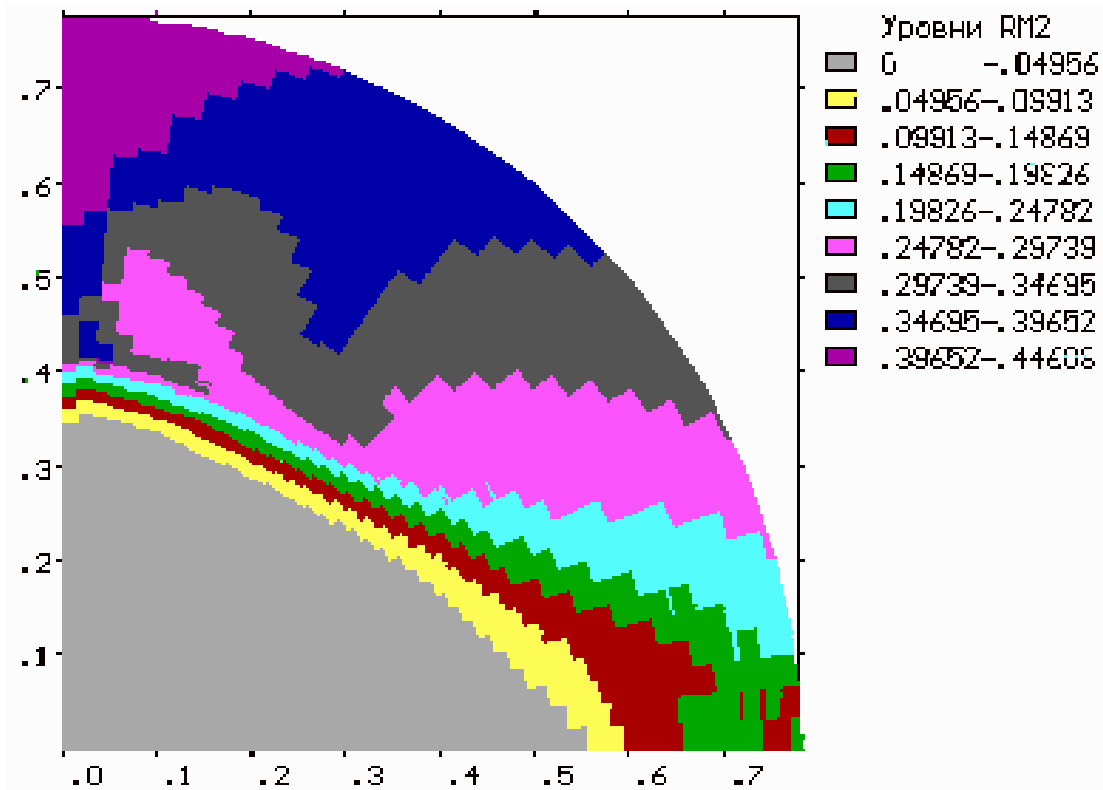
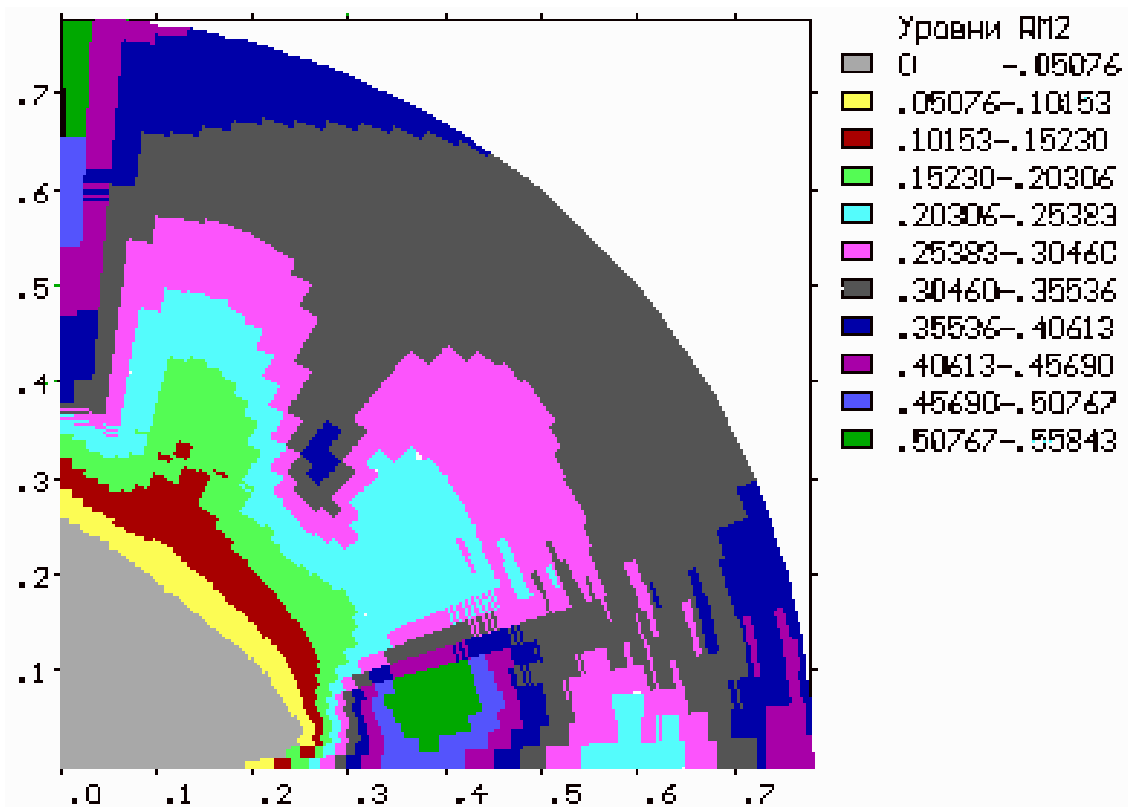fig8. Plot of the "magnetic" pressure, 2D task, P_m=0.44, t = 0.4

fig9. Plot of the "magnetic" pressure, 2D task, P_m=0.44, t = 0.6