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# Quantitative Theory for the Nonlinear Growth Rates in Richtmyer–Meshkov Instability*

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**Abstract.** We present a nonlinear theory of the unstable growth of fingers at material interfaces accelerated by shock waves in two and three dimensions. Our theory provides quantitative predictions for the overall growth rate of the unstable interface and the growth rates of spike and bubble in compressible fluids. Our theoretical predictions are in remarkable agreement with the results of full numerical simulations from early to late times, and agree, for the first time, with the experimental data on air-$SF_6$. Previous theoretical predictions of the growth rate for air-$SF_6$ unstable interfaces were about two times larger than the experimental data.

In 1960, Richtmyer showed theoretically that a perturbed material interface between two fluids of different density accelerated by a shock is unstable [1]. Ten years later, Meshkov performed experiments to confirm the Richtmyer's prediction [2]. This instability is known as Richtmyer-Meshkov (RM) instability and plays an important role in studies of supernova and inertial confinement fusion (ICF). Since then several experiments [3, 4] and numerical simulations [5]-[12] on the growth of the RM unstable interfaces have been performed. Several theories have been developed by different approaches [1], [13]-[21]. Most of previous theoretical work focused on the growth rate in linear regime. However, the growth of the RM unstable interface is nonlinear [8] and, for a long time, theories failed to give a quantitatively correct prediction for the growth rate of RM unstable interface in the nonlinear regime. Previous theoretical predictions were about twice as large as the experimental data on air-$SF_6$.

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In this paper, we report recent significant theoretical progresses made by the authors. We have developed non-linear theories for Richtmyer-Meshkov instability for compressible fluids in the case when the reflected wave is a shock. Our theories provide analytic predictions for the nonlinear growth rate of the overall unstable interface and the growth rates of spike and bubble. A spike is a portion of heavy fluid penetrating into light fluid, and a bubble is a portion of light fluid penetrating into heavy fluid. The overall growth rate of RM unstable interface is the growth rate of the half of the overall size of the mixing zone between the two fluids. The agreements among our theoretical predictions, the results of full nonlinear numerical simulations and experimental data are remarkable.

Richtmyer's impulsive model is a widely used theoretical model for the growth rate of RM unstable interfaces [1]. It predicts

$$v^{imp} = -\Delta u \frac{\rho' - \rho}{\rho' + \rho} k a_0(0+). \tag{1}$$

Here $\Delta u$ is the difference between the shocked and unshocked mean interface velocities. $\rho$ and $\rho'$ are the post-shocked fluid densities, and $a_0(0+)$ is the post-shocked perturbation amplitude at the interface. Richtmyer showed three examples in which the predictions of the impulsive model agree quite well with the asymptotic solutions of the linear theory [1]. A more extensive comparison over a large parameter space showed the domains of agreement and disagreement [16]. Even when the prediction of the impulsive model agrees with the result of the linear theory, it agrees in the regime where the nonlinearity is important and the linear theory is no longer valid [8].

In order to develop an approximate nonlinear theory for compressible RM unstable interface, we adopt the physical picture that the dominant effects of the compressibility occur near the shocks. This influences the material interface when the shocks are in the vicinity of the material interface, namely at early times. We assume that the initial disturbance at the interface is small. Then, at early times the compressibility is important and the nonlinearity is less important. As time evolves, the magnitude of the disturbance at the material interface increases significantly and the transmitted shock and reflected wave move away from the interface. The effects of compressibility are reduced and the nonlinearity starts to play a dominant role in the interfacial dynamics. From this physical picture, we see that at early times the dynamics of the system are mainly governed by the linearized Euler equations for compressible fluids, while at later times the dynamics are mainly governed by the nonlinear equations for incompressible fluids. The RM unstable system goes through a transition from a linear and compressible one at early times to a nonlinear and incompressible one at later times. Our approach is to qualitatively separate the dynamics of the RM instability into two stages corresponding to early and later times. We determine an approximate solution in each stage. Then we match the early time solution and the later time solution to obtain an analytical expression which changes gradually from one to the other. The matched solutions give quantitative predictions for the overall growth rate and the growth rates of spike and bubble in compressible RM instability from early to later times.

At early times, the dynamics are compressible and approximately linear. The solution is governed by the linearized Euler equations and can be found in [1] and [16].

At later times, the dynamics are nonlinear and approximately incompressible. The governing equations for inviscid, irrotational and incompressible fluids are

$$\begin{aligned} \Delta\phi(x,z,t) &= 0, \quad \text{in material 1,} \\ \Delta\phi'(x,z,t) &= 0, \quad \text{in material 2,} \end{aligned} \tag{2}$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = \eta, \tag{3}$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi'}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi'}{\partial z} = 0 \qquad \text{at } z = \eta, \tag{4}$$

$$-\rho'\frac{\partial \phi'}{\partial t} + \rho\frac{\partial \phi}{\partial t} + \frac{\rho'}{2}(\nabla\phi')^2 - \frac{\rho}{2}(\nabla\phi)^2 = 0 \qquad \text{at } z = \eta. \tag{5}$$

Here $\phi$ and $\phi'$ are the velocity potentials of the two fluids. $z = \eta(x,t)$ is the position of the interface at time $t$. The initial shape of the material interface is given by $\eta(x,0) = a_0\cos(kx)$.

Recently the authors have developed a procedure for solving (2)-(5) [19]. The method is a perturbation expansion of the solution in terms of $a_0$ and the solution procedure is recursive. The $n-$th order solutions are expressed in terms of the solutions of orders less than $n$. The explicit solution of the impulsive model through fourth order can be found in [19]. To be consistent with the early time linear solution which is single mode, we choose single mode initial growth rate $\dot{\eta}(x,0) = v_0\cos(kx)$. Following the solution procedure developed in [19]. the four leading order terms of $\eta$ have been derived explicitly. The results are

$$\begin{aligned} \eta^{(1)} &= (a_0 + v_0 t)\cos(kx), \\ \eta^{(2)} &= \frac{1}{2}Akv_0^2t^2\cos(2kx), \end{aligned}$$

$$\begin{aligned} \eta^{(3)} = &-\frac{1}{24}k^2v_0^2[(4A^2+1)v_0t^3 + 3a_0t^2]\cos(kx) \\ &+\frac{1}{8}k^2v_0^2[(4A^2-1)v_0t^3 - 3a_0t^2]\cos(3kx), \end{aligned}$$

$$\begin{aligned} \eta^{(4)} = &-\frac{1}{12}k^3v_0^2[4A^3v_0^2t^4 + 3Aa_0^2t^2]\cos(2kx) \\ &+\frac{1}{12}k^3v_0^2[(8A^3-4A)v_0^2t^4 - 8Av_0t^3 + 3Aa_0^2t^2]\cos(4kx). \end{aligned}$$

We then carry through the following three steps: (i) Construction of the series solutions for the overall growth rate and the growth rates of spike and bubble; (ii) Application of Pade approximation to construct an approximate solutions at later times; (iii) Construction of approximate solutions from early to later times for compressible fluids by matching the early time solution and later time solution. The matching determines $v_0$ as $v_{lin}$, the linear growth rate determined from the linear theory [1, 16] Following these three steps, we have [18, 20]

$$v = \frac{v_{lin}}{1 + v_{lin} a_0 k^2 t + \max\{0, a_0^2 k^2 - A^2 + \frac{1}{2}\} v_{lin}^2 k^2 t^2} \tag{6}$$

for the overall growth rate,

$$v_{sp} = v + \frac{A k v_{lin}^2 t}{1 + 2k^2 a_0 v_{lin} t + 4k^2 v_{lin}^2 [a_0^2 k^2 + \frac{1}{3}(1 - A^2)] t^2} \tag{7}$$

for the growth rate of the spike and

$$v_{bb} = -v + \frac{A k v_{lin}^2 t}{1 + 2k^2 a_0 v_{lin} t + 4k^2 v_{lin}^2 [a_0^2 k^2 + \frac{1}{3}(1 - A^2)] t^2} \tag{8}$$

for the growth rate of the bubble. Here $a_0$ is the post-shocked perturbation amplitude and $A = (\rho - \rho')/(\rho + \rho')$ is the post-shocked Atwood number.

It is easy to see that in the early time, or small amplitude limits, (6)-(8) approach to $v_{lin}$. Equations (6)-(8) show that all three growth rates decay at later times and that the spike grows faster than the bubble. We comment that our nonlinear theories given by (6)-(8) contain no adjustable parameter. Equations (6)-(8) are applicable to the systems with no indirect phase inversion only. An indirect phase inversion is defined for the situation $a_0(0+) v_{lin}(t \to \infty) < 0$. For the case of reflected shock, the indirect phase inversion usually does not occur [16].

Now we compare our theoretical predictions with the results from full numerical simulations. In Figure 1, we consider an air-$SF_6$ interface. A weak shock of Mach number 1.2 propagates from air to $SF_6$. The reflected wave is a shock. $a(0-) = 2.4$mm and $A(0+) = 0.701$. The wave length of the perturbation is 37.5mm and the pressure ahead of the shock is 0.8 bar. These parameters corresponds to Benjamin's experiments on air-$SF_6$ (a) is for the overall growth rate. (b) is for the overall amplitude determined by integrating (5) over time. (c) and (d) are for the growth rate of bubble and spike, respectively.

In Figure 2, we consider an Kr-Xe interface. The unstable interface is accelerated by a strong shock of Mach number 3.5 moving from Kr to Xe. The reflected wave is also a shock. $a(0-) = 5$mm and $A(0+) = 0.184$. The wave length is 36mm and the pressure ahead of the shock is 0.5 bar. These physical parameters correspond to

Zaytsev's experiments. The dimensionless perturbation amplitude $a_0(0-)k$ is 0.87. This amplitude is about two times as large as than the amplitude $a_0(0-)k = 0.40$ given in Figure 1 for the air-$SF_6$ case. (a) is for the overall growth rate. (b) is for the overall amplitude. (c) and (d) are for the growth rate of bubble and spike, respectively.

The results from linear compressible theory and from linear impulsive model given by (1) are also shown in Figures 1 and 2. Figures 1 and 2 show that our theoretical predictions are in the excellent agreement with the results from full numerical simulations, while the predictions of the linear theory for compressible fluids and the linear impulsive model are qualitatively incorrect at later times.

In experiments, it was difficult to measure the growth rate directly. Instead, one measured the amplitude of the disturbed interfaces, i.e. the half of the longitudinal distance between the spike and bubble tips. One assumed that the amplitude was a linear function of time and applied a linear regression analysis to determine the overall growth rate of the unstable interfaces. The overall growth rate determined from the experimental data was 9.2 m/s over the time period 310-750 $\mu s$, (see Figure 1(b)). When we applied the linear regression to the amplitude predicted by our theory and to the amplitude determined from numerical solution of full Euler equations, we found the identical results 9.3 m/s for the growth rate over that time period for both our theory and for the full numerical solutions. Therefore, the prediction of our theory is in excellent agreement with the experimental result, as well as with the full nonlinear numerical simulation. Predictions of the growth rate from the impulsive model and from the linear theory are 15.6 m/s and 16.0 m/s respectively.

Finally, we verify the physical picture on which the theory was based. In figure 3, we show predictions from linear compressible theory, nonlinear incompressible theory and nonlinear compressible theory, namely (6), as well as the result from full numerical simulations, for the overall growth rate of air-$SF_6$ interface. The nonlinear incompressible theory is obtained from setting $v_0$ to $v_{lin}^{\infty} = \lim_{t\to\infty} v_{lin}(t)$ in (6). The physical parameters here are identical to ones in Figure 1. Figure 3 verifies that our physical picture is reasonable. The dynamics of the RM unstable interface indeed changes from a linear compressible one at early times to a nonlinear incompressible one at later times.

We comment that one may replace $v_{lin}$ by its asymptotic limit $v_{lin}^{\infty} = \lim_{t\to\infty} v_{lin}(t)$ in (6)-(8) to obtain approximate expressions for non-linear growth rates at later times. For weak shocks, $v_{lin}^{\infty}$ can be approximated by the linear solution of the impulsive model given by (1).

We have extended our theoretical derivation to fluids in three dimensions [21]. The overall growth rate of the compressible RM unstable interface in three dimensions is

$$v = \frac{v_{lin}}{1 + v_{lin} a_0 k^2 \lambda_1 t + \max[0, a_0^2 k^2 \lambda_1^2 - 3\lambda_2] v_{lin}^2 k^2 t^2}. \quad (9)$$

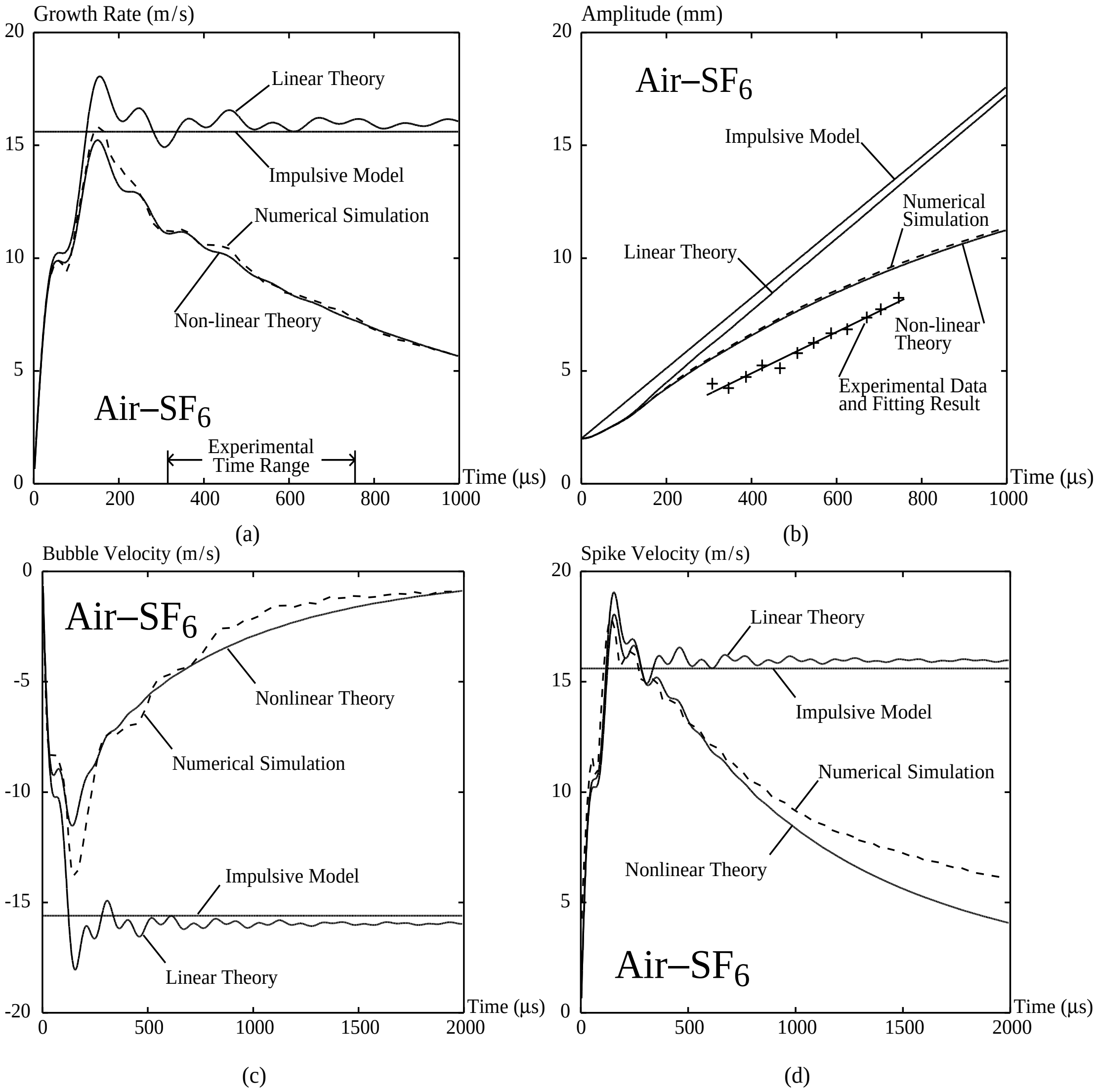


Figure 1: Comparison of predictions from linear theory, linear impulsive model, nonlinear theory and the results from full numerical simulation for air-$SF_6$ interface. A weak incident shock of Mach number 1.2 propagates from air to $SF_6$. Our theoretical predictions are in the excellent agreement with the results from full numerical simulations, while the predictions of the linear theory for compressible fluids and the linear impulsive model are qualitatively incorrect at later times. (a) is for the overall growth rate. (b) is for the overall amplitude. (c) and (d) are for the growth rate of bubble and spike, respectively.

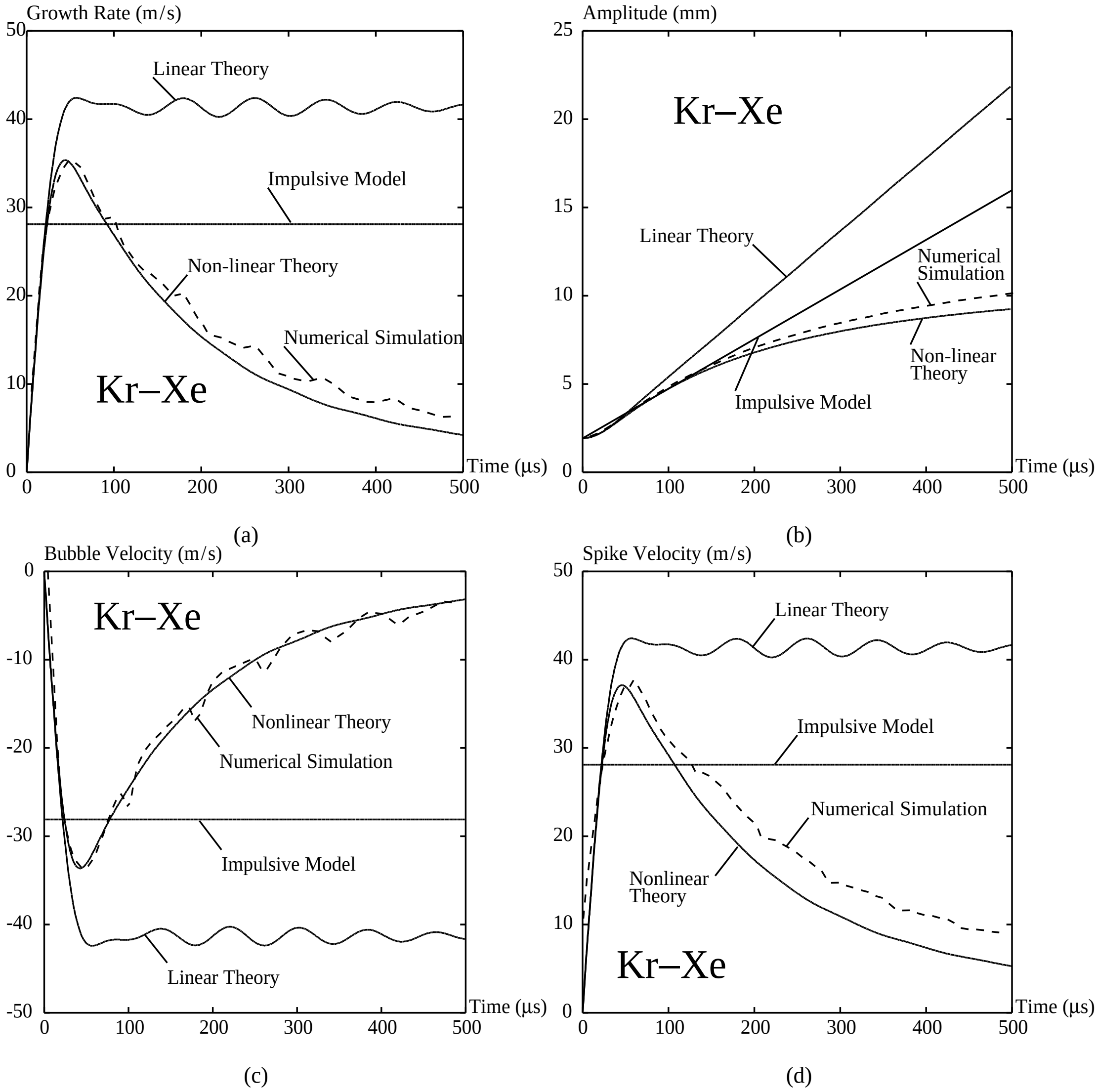


Figure 2: Comparison of predictions from linear theory, linear impulsive model, nonlinear theory and the results from full numerical simulation for Kr-Xe interface. A strong incident shock of Mach number 3.5 propagates from Kr to Xe. Our theoretical predictions are in the excellent agreement with the results from full numerical simulations, while the predictions of the linear theory for compressible fluids and the linear impulsive model are qualitatively incorrect at later times. (a) is for the overall growth rate. (b) is for the overall amplitude. (c) and (d) are for the growth rate of bubble and spike, respectively.

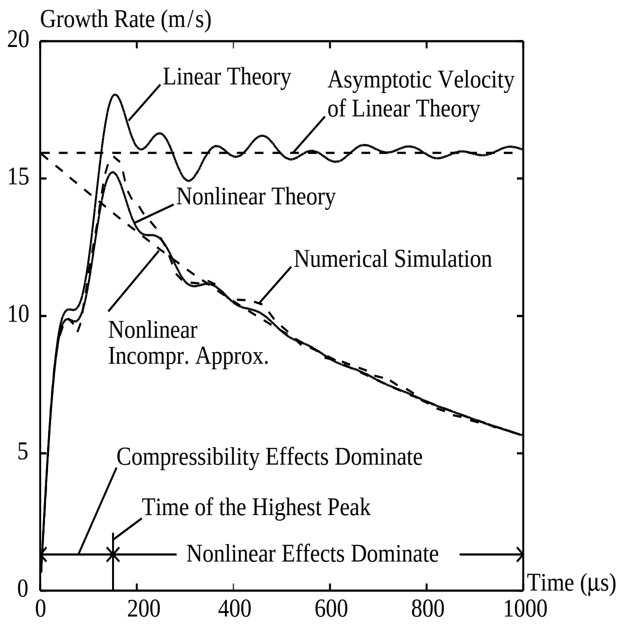


Figure 3: A comparison among the predictions from linear compressible theory, nonlinear incompressible theory and nonlinear compressible theory (8), as well as the result from full numerical simulations, for the overall growth rate of air-$SF_6$ unstable interface. Figure 3 shows that the dynamics of the RM unstable interface indeed change from a linear compressible linear one at early times to a nonlinear incompressible one at later times.

Here $v_{lin}$ is the growth rate of the linear theory in three dimensions. $k = \sqrt{k_x^2 + k_y^2}$ is total wave number of the initial perturbation at the material interface $\eta(x, y, t = 0) = a_0 \cos(k_x x) \cos(k_y y)$. $\lambda_1$ and $\lambda_2$ are dimensionless functions which depend on the post-shocked Atwood number $A$ and the $\theta$, orientation of the wave vector $(k_x, k_y)$. The explicit expressions of $\lambda_1$ and $\lambda_2$ can be found in [21]. See [21] for the quantitative predictions for the overall growth rate in three dimensions. It is found that for fixed total wave number $k$ and fixed Atwood number $A < 0.64$, the symmetric interface in three dimensions $(k_x = k_y)$ is most unstable, while the interface in two dimensions is least unstable. For the symmetric interface in three dimensions, the expressions for $\lambda_1$ and $\lambda_2$ are

$$\lambda_1 = \frac{1}{4}(2 - 5\sqrt{2} + 4\sqrt{5} - \sqrt{10})A^2 + \frac{1}{4}(4 + 7\sqrt{2} - 6\sqrt{5} + \sqrt{10}),$$
$$\lambda_2 = -\frac{1}{8}(7 + 7\sqrt{2} - 9\sqrt{5} + 3\sqrt{10})A^2 + \frac{1}{8}(4 + 7\sqrt{2} - 6\sqrt{5} + \sqrt{10}).$$

The predictions for the growth rates of spike and bubble in three dimensions are also given in [21].

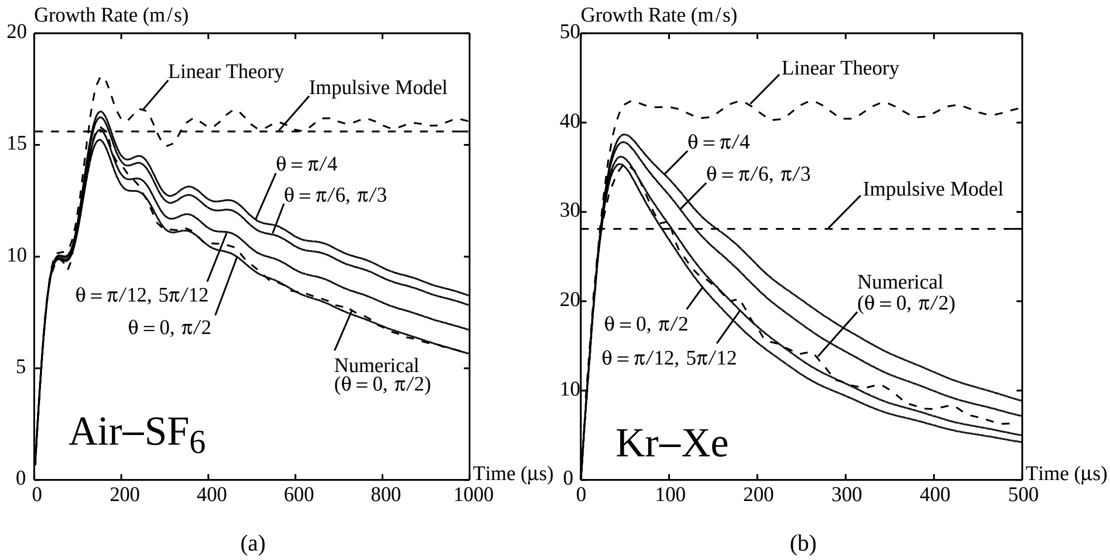


Figure 4: A comparison of the predictions of the overall growth rates from linear theory, impulsive model and nonlinear theory (9) in three dimensions for several different values of $\theta$ with fixed total wave number $k$. For comparison, the results of a full non-linear numerical simulation in two dimensions are also shown. (a) is for air-$SF_6$ case. (b) is for Kr-Xe case.

In Figure 4, we show the overall growth rates of (a)air-$SF_6$ and (b)Kr-Xe unstable interfaces for several different values of angle $\theta$ with fixed total wave number $k$. The physical parameters are the save as the ones given in Figures 1 and 2, except that the wave number $k$ should be interpreted as the total wave number $k = \sqrt{k_x^2 + k_y^2}$.

In conclusion, we have developed a quantitative nonlinear theory for the overall of growth rate as well as the growth rates of spike and bubble in compressible RM instability in two and three dimensions. Our theoretical predictions are in excellent agreement with the results from full numerical simulations. Our results show that at later times, nonlinearity is far more important than the compressibility.

## References


[1] R. D. Richtmyer, *Comm. Pure Appl. Math.* **13**, 297, (1960).

[2] E. E. Meshkov, NASA Tech. Trans. F-13, 074, (1970).

[3] A. N. Aleshin, E. V. Lazareva, S. G. Zaitsev, V. B. Rozanov, E. G. Gamalii and I. G. Lebo, *Sov. Phys. Dokl.* **35**, 159, (1990).

[4] R. Benjamin, D. Besnard, and J. Haas, LANL report LA-UR 92-1185, (1993).

[5] K. A. Meyer and P. J. Blewett, *Phys. Fluids* **15**, 753, (1972).

[6] L. D. Cloutman, and M. F. Wehner, *Phys. Fluids A* **4**, 1821, (1992).

[7] T. Pham and D. I. Meiron, *Phys. Fluids A* **5**, 344, (1993).

[8] J. Grove, R. Holmes, D. H. Sharp, Y. Yang and Q. Zhang, *Phys. Rev. Lett.* **71**, 3473, (1993).

[9] K. O. Mikaelian, *Phys. Rev. Lett.* **71**, 2903, (1993).

[10] R. L. Holmes, J. W. Grove and D. H. Sharp, *J. Fluid Mech.* **301**, 51, (1995).

[11] U. Alon, J. Hecht, D. Ofer and D. Shvarts, *Phys. Rev. Lett.* **74**, 534, (1995).

[12] D. L. Youngs, *Laser and Particle Beams* **14**, 725, (1994).

[13] G. Fraley, *Phys. Fluids* **29**, 376, (1986).

[14] S. W. Haan, *Phys. Fluids B* **3**, 2349, (1991).

[15] R. Samtaney and N. J. Zabusky, *Phys. Fluids A* **5**, 1285, (1993).

[16] Y. Yang, Q. Zhang and D. H. Sharp, *Phys. Fluids A* **6**, 1856, (1994).

[17] J. Hecht, U. Alon and D. Shvarts, *Phys. Fluids* **6**, 4019, (1994).

[18] Q. Zhang and S.-I. Sohn, *Phys. Lett. A* **212**, 149, (1996).

[19] Q. Zhang and S.-I. Sohn, Nonlinear Solutions of Unstable Fluid Mixing Driven by Impulsive Force, submitted to *Phys. Fluids.*

[20] Q. Zhang and S.-I. Sohn, Spike and Bubble Dynamics in Richtmyer-Meshkov Instability, submitted to *Phys. Rev. E.*

[21] Q. Zhang and S.-I. Sohn, Quantitative Theory of Richtmyer-Meshkov Instability in Three Dimensions, Preprint, University at Stony Brook.