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Turbulent Diffusion in Compressible Media

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Abstract. The diffusion of some admixture in compressible media showing an homogeneous,isotropic and stationary turbulence is considered. The turbulent diffusivities D_T are calculated from the exact numerical solution of nonlinear DIA-equation for the averaged Green's function for the ensemble of chaotic shock waves in infinite medium. It was studied the dependence of D_T on the degree of compressibility,turbulent Strouchal number and space-scales of turbulent motions.

1 DIA equation for compressible turbulence

Since the averaged number density $\langle n(\mathbf{r},t)\rangle$ and the fluctuating part $n_1(\mathbf{r},t)(\langle n_1(\mathbf{r},t)\rangle = 0)$ depend on each other, the **separate** equation for the averaged Green's function $\langle G(1,2)\rangle \equiv G(1-2) \equiv G(\mathbf{R},\tau)$ is nonlinear one and takes the form of hierarchy of nonlinear equations with increasing degree of nonlinearity. The first equation of this hierarchy with the second order nonlinearity is called the equation in direct interaction approximation (DIA). It has the form ($dn = d\mathbf{r}_n dt_n$, etc.):

$$G(1-2) = G_m(1-2) + \int d3 \int d4 G_m(1-3)\nabla_i^{(3)}\langle u_i(3)G(3-4)\nabla_j^{(4)}u_j(4)\rangle G(4-2) \quad (1)$$

Here $u_i(1) \equiv u_i(\mathbf{r}_1,t_1)$ is the turbulent velocity of the basic gas, $\langle \mathbf{u}\rangle = \mathbf{0}$, and $G_m(1-2) \equiv G_m(R,\tau)$ is the molecular Green's function with the molecular diffusivity D_m; $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2, \tau = t_1 - t_2$. It was shown by R.Kraichnan [4] for incompressible turbulence that the DIA-equation describes the turbulent diffusion very satisfactory. Therefore,it is natural to use eq. (1) also in the case of compressible medium. The Fourier transform of two-point velocity correlator $\langle u_n(1)u_m(2)\rangle \equiv B_{nm}(\mathbf{R},\tau)$ for compressible turbulence without helicity has the form (Batchelor [1]):

$$B_{nm}(\mathbf{p},\tau) = (\delta_{nm}p^2 - p_n p_m)f(p,\tau) + p_n p_m W(p,\tau) \quad (2)$$

The first term coincides with the correlator in the case of incompressible medium (div$\mathbf{u}$= 0) and is characterized by the generalized spectrum $E_{inc}(p,\tau) = \pi^{-2}p^4 f(p,\tau)$. The second term describes the compressible motions ($rot\mathbf{u} = 0$) and is characterized by the spectrum $E_{compr}(p,\tau) = \pi^{-2}p^4 W(p,\tau)/2$. The total spectrum is the sum of these independent spectra. The Fourier transform of eq. (1) in variable $\mathbf{R}$ depends on these spectra. The turbulent diffusivity D_T is related with the transformed Green's function $G(p,\tau)$ (Dolginov and Silant'ev [2]):

$$D_T^{(0)} = (1/3)\int_0^\infty dp \int_0^\infty d\tau [E_{inc}(p,\tau) - p\ \partial E_{compr}(p,\tau)/\partial p]\ G(p,\tau) \tag{3}$$

The first term in (3) is always positive and the second one may be negative. So, in general,the compressibility decreases the turbulent diffusivity. The qualitative discussion of this phenomenon was given by Silant'ev [5]. It should be stressed that due to nonlinearity of eq. (1) the expression (3) give the contribution to D_T of all the forms and degrees of the two-point velocity correlators. The contribution of remaining four-order irreducible correlators, i.e. of the form $\langle u_i(1)u_n(3)\rangle\langle u_j(2)u_m(4)\rangle$ with $t_1 > t_2 > t_3 > t_4$, grows from zero for $\xi = u_*\tau_0 p_0 = 0$ up to value of the order 10% for frozen turbulence having $\xi \gg 1$ (Silant'ev [6]). Here $u_*^2 = \langle u^2(\mathbf{r},t)\rangle$, τ_0 and $p_0 = 1/R_0$ are characteristic life-time and wave number of turbulent motions. Such good efficiency of DIA expression (3) is quite natural because the main contribution to turbulent diffusivity D_T give the most large-scale motions which are described by two-point velocity correlators. Four-point and higher correlators describe the fine structure of turbulence with small space-scales which give relatively small contribution to D_T.

2 Simple model of turbulence

The generalized spectra $E_{inc}(p,\tau)$ and $E_{compr}(p,\tau)$ are presented in (1) and (3) as integrated quantities. For this reason,the behavior of the Green's function and turbulent diffusivity D_T depend mostly on the values of dimensionless parameters $\xi = u_*\tau_0 p_0$ and other parameters,such as p_0/p_1 ,τ_0/τ_1 etc. In any case, the qualitative behavior of D_T as function of these parameters may be studied using the most simple model of chaotic compressible turbulent motions:

$$\begin{aligned} E(p,\tau) &= E_{inc}(p,\tau) + E_{compr}(p,\tau) \\ &= u_*^2[a\delta(p-p_0)exp(-\tau/\tau_0) + (1-a)\delta(p-p_1)exp(-\tau/\tau_1)] \end{aligned} \tag{4}$$

Here $u_*^2 = \langle \mathbf{u}^2(\mathbf{r},t)\rangle$, a is the part of turbulent energy involving into solenoidal (incompressible) motions. We shall use also the notations $u_0^2 = au_*^2$, $u_1^2 = (1-a)u_*^2$, so $u_*^2 = u_0^2 + u_1^2$. The model (4) may be considered as representing the ensemble of chaotic shock waves. We do not consider the ensemble of acoustic waves where D_T is extremely small, of the order of kinematic viscosity. The time of statistical relaxation τ_1 of compressible chaotic motions is directly related with the relative change of the volume element and may be estimated from the expression:

$$\langle(\Delta V/V\tau_1)^2\rangle \simeq \langle div^2\mathbf{u}(\mathbf{r},t)\rangle = \int_0^\infty dp\, p^2 E_{compr}(p,0) \simeq u_1^2 p_1^2 \qquad (5)$$

Qualitatively such relation is natural. Indeed, during the process of compression or decompression of a gas there exists a distinct correlation of velocities in some fixed points of space. So, we have the estimation $\tau_1 \simeq (\Delta V/V)/u_1 p_1$. This means that the Strouchal number $\xi_1 = u_1\tau_1 p_1 \simeq \Delta V/V$. The mean change of the volume element $\Delta V/V$ depends on particular physical processes which take place in medium. It is important that this parameter is restricted. So, for adiabatic motions one has (Katz [3]) $\Delta V/V < 2\gamma/(1+\gamma)$, with γ being the adiabatic constant. Thus, for mono atomic gas ($\gamma = 5/3$) one has $\xi_1 < 5/4$, and for two atomic - $\xi_1 < 7/6$. This means that the regime of frozen turbulence ($\xi_1 \gg 1$) does not exist for pure compressible (potential) motions. This is due to the fact that in free unbounded space the compressible chaotic motions of type some eddy with many numbers of revolution do not exist. Every act of compression and decompression looks like pure volume effect.

3 Results of calculation

First of all let us consider two limiting cases - pure incompressible ($a = 1$) and pure compressible ($a = 0$) turbulence. In these cases the dimensionless diffusivities $\underline{D}(\xi_0) = (u_0/p_0)^{-1}D_T$ and $\underline{D}(\xi_1) = (u_1/p_1)^{-1}D_T$ depend on one dimensionless parameter ξ_0 or $\xi_1 \simeq \Delta V/V$,respectively. For these limiting cases we have calculated also the contribution $D_T^{(1)}$ to turbulent diffusivity of four-order velocity correlators (by assumption that the ensemble of realizations is Gaussian). For small values of ξ_0 or ξ_1 the diffusivities (3) increase linearly : $\underline{D}_T(\xi_0) \simeq \xi_0/3$ and $\underline{D}_T(\xi_1) \simeq \xi_1/3$. Linear increasing occurs up to $\xi_0 \simeq 1$ for incompressible turbulence and up to $\xi_1 \simeq 0.5$ for pure compressible one. But later on the diffusivities $\underline{D}_T^{(0)}(\xi_0)$ and $\underline{D}_T^{(0)}(\xi_1)$ differ drastically. The $\underline{D}_T^{(0)}(\xi_0)$ curve grows monotonic and near $\xi_0 \simeq 5-10$ reaches its maximum value 0.622 (this is the limit of frozen turbulence).The contribution of four-order velocity correlators grows from zero at $\xi_0 = 0$ up to value (-0.069) for $\xi_0 \gg 1$. The $\underline{D}_T^{(0)}(\xi_1)$-curve grows up to $\xi_1 = 1.4$, where $\underline{D}_T^{(0)} = 0.235$, and then begin decrease. At $\xi_1 \simeq 3.7$ it is equal to zero. For $\xi_1 > 3.7$ the solution of DIA equation (1) gives the negative values for turbulent diffusivity. The contribution of four-order velocity correlators changes the situation. For $\xi_1 < 3.7$ the $\underline{D}_T^{(1)}(\xi_1)$ is negative and decreases the value of total diffusivity ($\underline{D}_{Tmax} = 0.169$ at $\xi_1 = 0.9$). At $\xi_1 = 3.7$ the total diffusivity is practically equal to zero ($\underline{D}_T = \underline{D}_T^{(0)} + \underline{D}_T^{(1)} \simeq 0.002$), but then the total diffusivity increases very rapidly (at $\xi_1 = 9$ one has $\underline{D}_T = 0.815$). The $D_T^{(1)}(\xi_1)$ may be considered as a small correction to DIA-value of D_T only up to $\xi_1 \simeq \Delta V/V \ll 1$ where it gives 25% of DIA value. The solutions of DIA-equation (1) become unstable for $\xi_1 > 10$. It seems that the real turbulent diffusivities $D_T(\xi_1)$ correspond to the values of parameter $\xi_1 \simeq \Delta V/V \leq 1$ where DIA values of D_T with correction $D_T^{(1)}$ give quite satisfactory exactness. For intermediate case it were calculated the dimensionless diffusivities $\underline{D}_T^{(0)}(\xi, a, \eta, \Delta V/V) = (u_*/p_0)^{-1} D_T^{(0)}$ for the values $a = 0, 0.1, ...1$, $\eta = p_0/p_1 = 0.5, 1, 2, 5$ and $\Delta V/V \equiv \xi_1 = 1, 1/3$. The basic parameter $\xi = u_* \tau_0 p_0$ was taken in the interval $0 < \xi < 10$. It was found that $\underline{D}_T^{(0)}(x, a, \eta, \Delta V/V)$ depends strongly on all the parameters. The obtained results allow to estimate the turbulent diffusivities in very large interval of parameters. For $\xi \longrightarrow 0$ the diffusivity due to incompressible motions tends to zero linearly and the total diffusivity is determined by compressible motions only. Therefore, one has

$$\underline{D}_T(\xi = 0) = (p_0/p_1)\sqrt{(1-a)}\underline{D}_{Tcompr}(\xi_1 = \Delta V/V) \tag{6}$$

The values of $\underline{D}_{Tcompr}(\xi)$ are due to pure compressible motions. For considered cases $\Delta V/V = 1$ and $1/3$ one has $\underline{D}_{Tcompr}^{(0)}(1) = 0.22317$ and $\underline{D}_{Tcompr}^{(0)}(1/3) = 0.10528$. For larger values of parameter ξ the total diffusivity is determined by both types of motions. The calculations show that this diffusivity is less than the sum of diffusivities of compressible and incompressible motions considered as independent ones. This is very natural. Indeed,the existence of two types of motions makes the turbulence less regular than by the acting some one of these motions and,as a result,the turbulent diffusivity decreases. Let us give one characteristic example going out from our calculations: for $a = 0.5$, $\xi = 10$ and $\eta = 1$, $\Delta V/V = 1$ the decreasing consists of 25%.If $\eta = 2$ this difference is greater $\simeq 35\%$. According to (6) for $p_0/p_1 > 3$ and $\Delta V/V = 1$ the constant ($a = 0$) diffusivity due to compressible motions will exceed the maximum value ($D_T^{(0)} = 0.622 u_0/p_0$) of diffusivity due to incompressible motions.

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