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Stability of Point Blast Wave in Gas with Density Having Power Dependence on Radius

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1 Introduction

The instability of point (Sedov) blast wave propagating through a uniform perfect gas with low adiabatic index γ ($\gamma < \gamma_0 = 1.20$) was shown earlier theoretically in [1, 2, 3]. Later on [4] the experimental proof of this result was obtained so there are reasons to apply technique improved in [1] to other similar cases. The case of blast wave in non-uniform density gas (initial density ρ_0 is a power function of radius ($\rho_0 \sim r^k$) is considered in this paper. According to [1] stability is considered in respect to small (linear) perturbations expanded in spherical harmonics $Y_{nm}(\chi, \varphi)$, components of expansion being represented in self-similar form (only the case of $m = 0$ is considered):

$$f(t, \xi, \chi) = \sum_{n=0}^{\infty} t^{\lambda} f_n(z) \times Y_n(\chi), \tag{1}$$

where $\lambda = \lambda(n)$ - complex index, ξ - Lagrange radius, R - shock wave front radius ($R \sim t^{\frac{2}{k+5}}$), $z = \xi/R$ - self-similar argument.

The considered self-similar functions describe an asymptotic mode of perturbations development corresponding to large time values (much higher than those needed for sound wave crossing an explosion area). In this case :

- perturbations are independent on initial conditions;

- front values perturbations (for example, front radius perturbation R_1 have power time dependence ($R_1 \sim t^{(\lambda+1)\frac{2}{k+5}}$ or $\frac{R_1}{R} \sim R^{\lambda}$).

It is natural to suggest that pressure in the explosion center has no divergences and hence [1, 2] pressure perturbations are zero in the center.

2 System of self-similar equations for perturbations

According to [1] we derive linear equations for perturbations using linearization of hydrodynamic equations (2) written in Lagrangian form.

$$\frac{\partial}{\partial t}(P/\rho^{\gamma}) = 0 \tag{2}$$

$$\frac{\rho_{00}}{\rho} = \frac{D(r^3, \cos\theta, \varphi)}{D(\xi^3, \cos\chi, \psi)} \tag{3}$$

$$\frac{1}{\rho_{00}}\frac{\partial P}{\partial \eta_i} + \sum_{j=1}^{3} \frac{\partial x_j}{\partial \eta_i}\frac{\partial^2 x_j}{\partial t^2} = 0 \tag{4}$$

where $i = 1, 2, 3$; t, $\eta_1 = \xi$, $\eta_2 = \chi$, and $\eta_3 = \psi$ stand for Lagrangian variables; $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$, $x_3 = r\cos\theta$, P, and ρ stand for unknown values describing three dimensional motion behind the shock front; $\rho_{00} = \rho_{00}(\xi) \sim \xi^k$ is the gas density before shock front.

Corresponding 1-d (spherical) self-similar solution of (2) is well-known [5]. It can be written with functions $x(z)$, $P(z)$, $\rho(z)$:

$$r_0(t,\xi) = Rx(z)$$

$$P_0(t,\xi) = \frac{2}{\gamma+1}\rho_{00}\dot{R}^2 P(z)$$

$$\rho_0(t,\xi) = \frac{\gamma+1}{\gamma-1}\rho_{00}\rho(z),$$

$$\theta_0 = \chi, \quad \varphi_0 = \psi$$

where x,P, ρ - satisfy to following system of equations:

$$\rho = \frac{\gamma-1}{\gamma+1}\frac{z^{k+3}}{wx^3}, \tag{5}$$

$$P = \rho^{\gamma}/z^{3+k\gamma}, \tag{6}$$

$$\frac{P'}{P} = -\gamma\frac{ab}{w}x(1) = P(1) = \rho(1) = 1$$

Here and below we have:

$$w = \frac{x'}{x} = \frac{zdx}{xdz}$$

$$b = \frac{\gamma+1}{2\gamma}\frac{wz^{k+3}}{Px}$$

$$a = w' + (w + \frac{k+3}{2})(w-1)$$

According to [1] we write variables in (2-4) in terms of small self-similar perturbations (here $\rho_{00} = \rho_{00}(t) \sim R^k$):

$$r(t,\xi,\chi) = r_0(t,\xi) + \frac{2}{\gamma+1}\sum_n R_{1n}(t)x_{1n}(z)Y_n(\chi) \tag{7}$$

$$P(t,\xi,\chi) = P_0(t,\xi) + \frac{2}{\gamma+1}\rho_{00}\frac{\dot{R}^2}{R}\sum_n R_{1n}(t)P_{1n}(z)Y_n(\chi) \tag{8}$$

$$\rho(t,\xi,\chi) = \rho_0(t,\xi) + \frac{\gamma+1}{\gamma-1}\rho_{00}\frac{1}{R}\sum_n R_{1n}(t)\rho_{1n}(z)Y_n(\chi) \tag{9}$$

$$\theta(t,\xi,\chi) = \chi + \frac{2}{\gamma+1}\frac{1}{R}\sum_n R_{1n}(t)\nu_{1n}(z)\frac{dY_n(\chi)}{d\chi}. \tag{10}$$

After linearization we get a system of ordinary differential equations for functions:

$$P_2(z) = \frac{xP_1}{z^{k+3+\lambda}}, \quad x_2(z) = \frac{x_1}{xz^{\lambda}}, \quad \nu_2(z) = \frac{\nu_1}{z^{\lambda}}$$

$$bP_2 + (\widehat{D} + 3w + \lambda)x_2 - n(n+1)w\nu_2 = \frac{\gamma+1}{2\gamma}(2\lambda + k + 5) \tag{11}$$

$$wP_2 + ax_2 - (\widehat{D} + \frac{k+1}{2})\widehat{D}\nu_2 = 0 \tag{12}$$

$$\begin{aligned}(\widehat{D} - w + k + 3 + \lambda)P_2 + ((\widehat{D} + \frac{k+1}{2})\widehat{D} - a)x_2 \\ +n(n+1)a\nu_2 = 0\end{aligned} \tag{13}$$

(where $\widehat{D}$ stands for differential operator:

$$(\widehat{D} + \frac{f'}{f})F(z) = \frac{1}{f}\frac{zd}{dz}(fF)$$

Boundary conditions on the shock front ($z = 1$) are set in a usual manner:

$$\nu_2(1) = 0, \quad \widehat{D}\nu_2(1) = 1, \quad x_2(1) = 1$$

$$P_2(1) = 2\lambda + \frac{k(\gamma+1) + 5 - 3\gamma}{(\gamma+1)^2} \tag{14}$$

System (12-14) has the 4th order so four boundary conditions (15) are the complete necessary set of boundary conditions. Hence one more (the fifth) condition specified on the other (center) edge:

$$P_1(0) = 0$$

reduces the problem of solving the system (12-14) to the eigenvalue problem. The problem is solved numerically, and complex exponents $\lambda(n)$ are calculated as eigenvalues.

3 Results discussion

The considered problem has an analytical solution in case of γ being close to unity. For $\gamma = 1$ eigenvalues $\lambda(n)$ satisfy a simple equation:

$$(\lambda + k + 3)^2 (\lambda + \frac{k+5}{2})^2 - \frac{(9+k)}{2}(\lambda + k + 3)(\lambda + \frac{k+5}{2}) + n(n+1)\frac{(k+3)^2}{4} = 0 \tag{15}$$

In general case ($\gamma > 1$) the problem should be solved numerically. The complex values of $\lambda(n)$ thus found and calculated for $\gamma = 1.2$ and $\gamma = 1.667$ in a wide range of n are presented in Tab.1 as functions of n for $k = -2, -1, 0, 1$. The function $\gamma_0(n)$ which is satisfying the condition $Re\lambda(\gamma_0(n), n) = 0$ is presented in Tab.2 This function shows the boundary of instability region in the $(n - \gamma)$ plane.

The stability condition is $Re\lambda < 0$ so the stability condition with respect to all self-similar perturbations is $\gamma > \gamma_{cr} = max(\gamma_0(n))$.

This critical value γ_{cr} of adiabatic exponent turns out to be a slight function of k. For k equal to -2, -1, 0, 1, 2 corresponding values of γ_{cr} are equal to 1.12, 1.17, 1.20, 1.21, 1.22.

Let us consider separately the case of the first harmonic number ($n = 1$). We have two eigenvalues $\lambda_1(1)$, $\lambda_2(1)$ in this case, both real and negative. The greatest (of the greatest modulus) of them does not depend on γ; it is determined with formula:

$$\lambda_2(1) = -\frac{k+3}{2},$$

and corresponds to explosion direct motion with momentum conservation.

References

[1] V. M. Ktitorov, Voprosy Atomnoi Nauki i Tehniki (Atomic Science and Technique Issues), Ser. Theor. Appl. Phys, 2, p.28 (1984).

[2] D. Ryu and E. T. Vishniac, Astrophys. J. 313, 820 (1987).

[3] E. T. Vishniac and D. Ryu, Astrophys. J. 337, 917 (1989).

[4] J. Grun, J. Stamper, C. Manca, et al., Phys. Rev. Let. 66,21,p.2738, (1991).

[5] V. P. Korobeinikov, N. S. Melnikova, E. V. Ryazanov, Point blast theory, Moscow, Fizmatgiz, 1961.

Harmonic number n	$k=-2$ Reλ	Imλ	$k=-1$ Reλ	Imλ	$k=0$ Reλ	Imλ	$k=1$ Reλ	Imλ
1	-0.3	0	-0.62	0	-1	0	-1.55	0
1	-0.5	0	-1	0	-1.5	0	-2	0
2	-0.24	0.64	-0.64	0.86	-1.04	1.03	-1.48	1.26
3	-0.29	0.89	-0.51	1.25	-0.86	1.5	-1.28	1.69
4	-0.27	1.14	-0.4	1.57	-0.71	1.88	-1.1	2.12
5	-0.28	1.38	-0.32	1.86	-0.58	2.21	-0.94	2.5
6	-0.3	1.61	-0.25	2.12	-0.47	2.52	-0.79	2.84
7	-0.34	1.84	-0.2	2.37	-0.38	2.8	-0.67	3.15
8	-0.4	2.08	-0.17	2.61	-0.3	3.07	-0.55	3.45
9	-0.48	2.31	-0.15	2.85	-0.23	3.32	-0.45	3.73
10	-0.57	2.56	-0.14	3.08	-0.17	3.57	-0.36	4
12	-0.81	3.1	-0.15	3.53	-0.09	4.06	-0.21	4.52
14	-1.06	3.76	-0.19	3.99	-0.04	4.52	-0.1	5.01
16	-1.22	4.49	-0.27	4.44	-0.02	4.98	-0.01	5.49
18	-1.32	5.21	-0.39	4.9	-0.02	5.43	0.049	5.96
20	-1.39	5.92	-0.53	5.38	-0.06	5.89	0.084	6.42
24	-1.49	7.28	-0.9	6.38	-0.19	6.79	0.089	7.33
28	-1.57	8.62	-1.35	7.52	-0.41	7.72	0.02	8.23
32	-1.64	9.95	-1.79	8.85	-0.69	8.68	-0.12	9.14
36	-1.69	11.3	-2.1	10.3	-1.06	9.69	-0.31	10.1
40	-1.74	12.6	-2.31	11.7	-1.48	10.8	-0.57	11

Table 1: Eigenvalues $\lambda(n, \gamma, k)$, $\gamma = 1.2$.

Harmonic number n	$k=-1$ Reλ	Imλ	$k=0$ Reλ	Imλ	$k=1$ Reλ	Imλ
1	-0.7	0	-1	0	-1.45	0
1	-1	0	-1.5	0	-2	0
2	-0.8	1.02	-1.13	1.24	-1.55	1.51
3	-0.84	1.55	-1.07	1.83	-1.45	2.06
4	-0.93	2.05	-1.07	2.33	-1.39	2.61
5	-1.04	2.56	-1.11	2.81	-1.37	3.11
6	-1.17	3.1	-1.19	3.29	-1.39	3.59
7	-1.29	3.68	-1.28	3.78	-1.44	4.06
8	-1.38	4.28	-1.39	4.28	-1.5	4.53
9	-1.44	4.88	-1.51	4.8	-1.59	5.02
10	-1.48	5.48	-1.63	5.34	-1.68	5.51
12	-1.51	6.65	-1.85	6.47	-1.9	6.53
14	-1.52	7.78	-2.02	7.65	-2.12	7.6
16	-1.53	8.88	-2.13	8.84	-2.33	8.72
18	-1.53	9.97	-2.2	10	-2.52	9.87
20	-1.53	11.1	-2.23	11.2	-2.67	11.1
24	-1.52	13.2	-2.27	13.5	-2.86	13.4
28	-1.52	15.3	-2.28	15.7	-2.95	15.7
32	-1.51	17.4	-2.28	17.8	-3	18
36	-1.51	19.5	-2.28	20	-3.02	20.3
40	-1.51	21.6	-2.28	22.1	-3.03	22.5

Table 2: Eigenvalues $\lambda(n,\gamma,k)$, $\gamma=1.667$.

Harmonic number n	$k = -2$	$k = -1$	$k = 0$	$k = 1$
4	1			
5	1.08			
6	1.10	1		
7	1.12	1.11		
8	1.12	1.14	1	
9	1.12	1.16	1.11	
10	1.12	1.17	1.15	1
11	1.12	1.17	1.17	1.11
12	1.11	1.17	1.18	1.14
13	1.11	1.17	1.19	1.17
14	1.11	1.17	1.20	1.18
15	1.11	1.17	1.20	1.19
16	1.10	1.16	1.20	1.19
17	1.10	1.16	1.20	1.20
18	1.10	1.16	1.20	1.21
20		1.16	1.20	1.21
22			1.19	1.21
24				1.21

Table 3: Adiabatic exponent values $\gamma_0(n,k)$ limiting the region of instability: $Re\lambda(\gamma_0,n,k)=0$; if $\gamma<\gamma_0$ then $Re\lambda(\gamma,n,k)>0$