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Lagrangian Treatment of the Rayleigh–Taylor Instability

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Abstract. A new formulation of the Lagrangian equations for the evolution of Rayleigh-Taylor instability in inviscid incompressible fluids is presented. A set of exact coupled mode equations which govern the evolution of the velocity field and the nonlinear motion of the surface is derived. Unlike the traditional Eulerian mode expansion, which requires an infinitely growing number of modes at the nonlinear stage, the present expansion converges very rapidly. The use of the formalism for analyzing the nonlinear stage of the instability is demonstrated by analytical and numerical solutions.

1 Introduction

The Rayleigh-Taylor (RT) and Richtmyer-Meshkov (RM) instabilities occur when a light fluid supports a heavier one against gravity, or pushes it in a constant acceleration (RT instability), or after a shock has passed through the interface between two fluids. (RM instability). Initially, random perturbations at the interface grow exponentially in time. In the nonlinear stage of the instability, the interface is strongly distorted. Round “bubbles” of light fluid enter the heavy fluid and narrow “spikes” of heavy fluid penetrate the lighter one. Eventually the “bubbles”-“spikes” structure brakes down and a turbulent mixing of the two fluids occurs. The present work concentrates on the linear and nonlinear stages of the instability before turbulence takes over.

Linear and nonlinear theories and as well as analyses of numerical simulations and experiments rely on the expansion of the interface and the velocity field in Fourier modes. It is well known, from theoretical work[2][3], that this expansion converges only at the very early stage when the bubbles amplitude is smaller than 10% of the wavelength.

Kull[3] attempted to solve, numerically, the nonlinear fluid equations by expanding the velocity potential and the interface in terms of Fourier modes. He has found that, during time evolution, the convergence of the series becomes increasingly worse until the "Fourier ansatz ceases to be an appropriate representation". The divergence of the expansion occurs as the bubbles amplitude approaches 10% of the wavelength.

In perturbation expansions[1],[2], the nonlinear coupled mode equations are solved by an iterative process starting with the linear solution. In view of Kull's results it is not surprising that also the perturbation expansion converges only at the very beginning of the nonlinear stage when the mode amplitude is smaller than 10% of its wavelength[2]. The Layzer model[4] does not encounter this difficulty, however, it is limited to the description of the near vicinity of the bubble's top.

The purpose of the present work was to find alternative theoretical framework and mode expansion which do not suffer from the shortcomings described above, and can describe also the nonlinear stage of the instability. In the rest of the manuscript I will show that Lagrangian formulation of the problem, naturally suggests the appropriate mode expansion, and that the resulting mode coupling equations may be solved by a perturbation expansion which converges very rapidly. The method may be viewed as an extension of Ott's treatment of RT instability in a thin Layer [5].

2 General formulation in three dimensions

Consider an incompressible fluid which is supported by a mass-less fluid. Denote the initial position of a fluid element by $\vec{r}_0$. At later times, the location of the element depends on $\vec{r}_0$. Denote the x, y, z components of $\vec{r}_0$ by ξ, η, α. For incompressible fluid, the Jacobian of the transformation from ξ, η, α to x, y, z is constant i.e.

$$(\vec{r}_\xi \times \vec{r}_\eta) \cdot \vec{r}_{\ a} = const. \tag{1}$$

(We denote the derivatives of $\vec{r}$ with respect to ξ, η, α and t by subscripts). For irrotational motion, the matrix $\vec{\nabla}\vec{v}$ (where $\vec{v}$ is the Eulerian velocity) is symmetric.Consequently, the relation $\vec{r}_\xi \cdot \vec{\nabla}\vec{v} \cdot \vec{r}_\eta - \vec{r}_\eta \cdot \vec{\nabla}\vec{v} \cdot \vec{r}_\xi = 0$ and similar relations for the ξ, α and η, α elements should hold. Using the chain rule for derivatives, these relations are converted to a set of three differential equations which involve only Lagrangian variables:

$$\vec{r}_{\xi t} \cdot \vec{r}_{\ \eta} - \vec{r}_{\ \eta t} \cdot \vec{r}_{\ \xi} = 0,\ \vec{r}_{\xi, t} \cdot \vec{r}_\alpha - \vec{r}_{\alpha t} \cdot \vec{r}_\xi = 0,\ \vec{r}_{\eta t} \cdot \vec{r}_{\ \alpha} - \vec{r}_{\ \alpha t} \cdot \vec{r}_{\ \eta} = 0 \tag{2}$$

The above equations 1,2 should be supplemented by boundary conditions. Assuming periodicity in the ξ,η variables and demanding that the fluid velocity will vanish as $\alpha \to \infty$ supply part of the conditions. We still have to determine the boundary conditions at the fluid surface. At $t = 0$ the surface is at $\vec{r} = (\xi, \eta, 0)$, denote its location at later times by $\vec{R}(\xi, \eta, t)$ (i.e. $\vec{R}(\xi, \eta, t) \equiv \vec{r}(\xi, \eta, 0, t)$). Multiplying the equation of motion of a fluid element which is at the surface; $\vec{R}_{tt} = \left\{ \vec{\nabla} P(\vec{r}, t) \right\}_{\vec{r}=\vec{R}(\xi,\eta,\alpha,t)} + g\hat{z}$, by $\vec{R}_\xi$ and $\vec{R}_\eta$, taking into account the fact that the pressure gradient is normal to the fluid surface and that the vectors $\vec{R}_\xi$ and $\vec{R}_\eta$ are tangential to the surface we get the boundary conditions:

$$\vec{R}_\xi \cdot \vec{R}_{tt} = g\vec{R}_\xi \cdot \hat{z}, \qquad \vec{R}_\eta \cdot \vec{R}_{tt} = g\vec{R}_\eta \cdot \hat{z} \tag{3}$$

(g is the gravitational acceleration).

Equations 1,,2 together with the boundary conditions at $\alpha = 0$ (equations 3) and at $\alpha \to \infty$, and the periodicity conditions in the ξ,η variables uniquely determine the motion of the fluid.

3 Coupled Mode Equations

In the following we shall limit our treatment to systems in which no motion occurs in the y direction. In this case: $\vec{r} = (x, \eta, z)$, and $\vec{r}_\xi = (x_\xi, 0, z_\xi)$.

We assume that the displacement of a fluid element $x(\xi, \alpha, t) - \xi$ and $z(\xi, \alpha, t) - \alpha$ may be expanded in a Fourier series, i.e.:$x(\xi, \alpha, t) = \xi + \sum_k X_k(\alpha, t)e^{ik\cdot\xi}$ and $z(\xi, \alpha, t) = \alpha + \sum_k Z_k(\alpha, t)e^{ik\cdot\xi}$ Using the expansion in equations 1,2 and combining them we get:

$$\dot{U}_k'(\alpha, t) - k\dot{U}_k(\alpha, t) = \sum_{k'} ik' \left(\dot{U}_{k'}(\alpha, t)V_{k-k'}'(\alpha, t) - V_{k'}(\alpha, t)\dot{U}_{k-k'}'(\alpha, t)\right) \tag{4}$$

and its complex conjugate. In the above equation $U_k(\alpha, t) \equiv X_k(\alpha, t) + iZ_k(\alpha, t)$ and $V_k(\alpha, t) \equiv X_k(\alpha, t) - iZ_k(\alpha, t)$ and the derivatives of Z_k U_k V_k and X_k with respect to t is denoted by by a dot and with respect to α by a prime. Similarly, equation 3, yields:

$$\begin{aligned} \ddot{X}_k(0, t) - gikZ_k(0, t) &= -\sum_{k'} i\,(k - k')\, X_{k-k'}(0, t)\ddot{X}_{k'}(0, t) \\ &\quad -\sum_{k'} i\,(k - k')\, Z_{k-k'}(0, t)\ddot{Z}_{k'}(0, t) \end{aligned} \tag{5}$$

We have to solve these equations with the requirement that, as $\alpha \to \infty$, the velocity field vanishes, i.e. $\lim_{\alpha\to\infty} \dot{X}_k(\alpha, t) = \lim_{\alpha\to\infty} \dot{Z}_k(\alpha, t) = \lim_{\alpha\to\infty} \dot{U}_k(\alpha, t) = \lim_{\alpha\to\infty} \dot{V}_k(\alpha, t) = 0$. In the following section I will present solutions of these equations in the linear and nonlinear regimes.

4 Solutions

The solution of equation 4, for $X_0(\alpha, t)$ and $Z_0(\alpha, t)$ is: $\dot{X}_0(\alpha, t) = 0$, $Z_0(\alpha, t) = -\sum_{k'} 2ik'X_{k'}(\alpha, t)Z_{-k'}(\alpha, t)$. An exact solution in the case $k \neq 0$ was not obtained, however linearizing equation 4 one can immediately see that a solution with vanishing $\dot{X}_k$, $\dot{Z}_k$ at $\alpha \to \infty$ is possible only if at $\alpha = 0$ (i.e. at the interface) $\dot{X}_k$ and $\dot{Z}_k$ are related by: $\dot{Z}_k(0,t) = i\frac{k}{|k|}\dot{Z}_k(0,t)$ (zero order). Using this result in equation 5 yields a reduced set of ordinary differential equations of the form

$$\ddot{X}_k(0,t) - \sum_{k'} \Gamma^2_{k,k'} X_{k'}(0,t) = 0 \tag{6}$$

where $\Gamma^2_{k,k'}$ depends on the amplitudes X_k and velocities $\dot{X}_k$ of all the modes in the system.

Note that equation 6 has the same form as the standard linear equation for RT instability, but with the linear growth rate replaced by an effective nonlinear "growth matrix" $\Gamma_{k,k'}$ which depends on the amplitudes and velocities of all the modes in the system. A detailed analysis shows that equation 6 does not allow generation of new modes in the system. For example an initial sinusoidal perturbation with an amplitude x_0 and velocity v_0 and a wave number k will evolve as a single Lagrangian mode i.e. ;$x(\xi,0,t) = 2X_k(0,t)\cos(k\xi)$, $z(\xi,0,t) = -2iZ_k(0,t)\sin(k\xi) - 2ikX_k(0,t)Z_k(0,t)$. Solving equation 6, using the relations between X_k and Z_k, substituting in the above relations for $x(\xi,0,t)$, $z(\xi,0,t)$ we can now plot the surface $z(x,t)$ at various times. In figure (1a) we show a case with gravitational acceleration $g = 1$. The system is initiated with a single mode with $k = 1$ $x_0 = 0.005$ and $v_0 = 0$. (plot (A) corresponds to $t = 4.2825$,and (B) to $t = 4.96$). At late times the surface has a falling down spike at $k\xi = \pi/2$ and a rising bubble with a tip at $k\xi = 3\pi/2$. Higher orders in the perturbation expansion are obtained by iterating the zero order result in equations 4. For example, in the first order, for a single mode, we get :$Z_k(0,t) = \frac{1}{3}i\frac{k}{|k|}X_k(0,t)\left[\frac{3-3kX_k(0,t)+k^2X_k^2(0,t)+k^3X_k^3(0,t)}{1-kX_k(0,t)-k^2X_k^2(0,t)+k^3X_k^3(0,t)}\right]$. As we shall see, the first order is sufficient for an accurate description of the bubble velocity at all times. In figure (1b), the bubble velocity is plotted as a function of time (line A). For comparison I have added on this plot also the bubble velocity predicted by the Layzer model [4] (line B) and by second and third order Eulerian perturbation expansion[1] (C and D), and the result obtained by using the zero order solution of equation 4, in equation 5 (E).

The Layzer model[4] is not capable of describing the whole interface, nevertheless its prediction for the velocity of the bubble's tip is known to be accurate. Note that, also the first order in the present approach (line (A) in figure (1b)) describes correctly the velocity of the bubble both in the linear stage and at saturation. More detailed results for more complicated cases may be obtained by a numerical solution of equations 4, 5, discretizing the variables α and t. In contrast to the conventional Eulerian mode expansion which requires an infinitely growing number of modes[3][2], in the present method a bubble-spike structure is described by few modes. Also, since the discretization of the equations in the direction parallel to the interface is obtained by mode expansion and

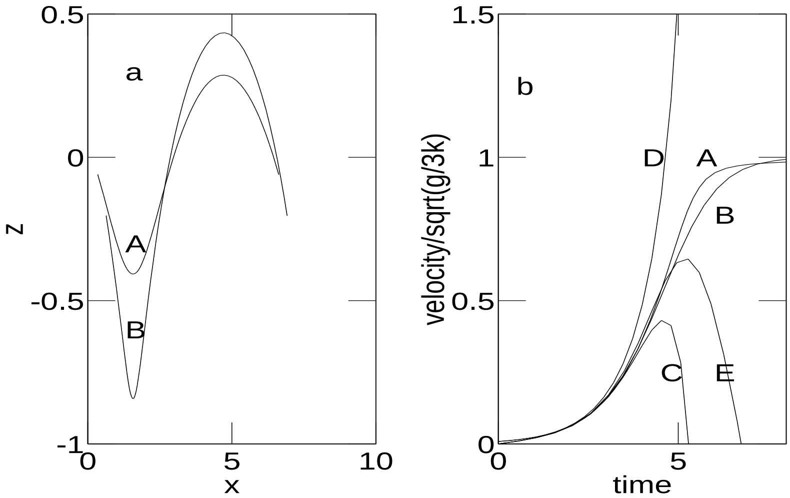

Figure 1:

not by a spatial grid, the numerical solution of equations 4, 5 does not suffer from the problem of mesh distortion which is typical of Lagrangian codes. These results indicate that the Lagrangian approach developed in the present work is a useful tool for the analysis of the nonlinear stage of the RT and RM instabilities.

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