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Self–Similar Evolution of Rayleigh-Taylor Instability in the Corner-Point Regions

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In studying Rayleigh-Taylor instability setting up the problem of periodic constant wavelength perturbations [1, 2, 3, 4] is in wide use. In this setting up the problem of instability evolution at the non-linear stage reduces to stationary at large times and this circumstance facilitates its analysis [2, 3].

Localized perturbations studied in [5, 6] are another type of perturbations whose evolution is universal in its character. A possible type of perturbations of no characteristic size are those in the form a dihedral angle which were studied in [6, 7] (as a special case of such a perturbation type for an angle equal to π, localized perturbations in a plane problem can be considered, i.e. perturbations in the form of a straight line plotted on fluid surface). The problem of evolution of such perturbations can be of interest both on its own for practical applications and as auxiliary for model construction.

The basic feature of evolution of perturbation in the form of a corner is non-linearity as the perturbations are not small (the perturbation amplitude is of the order of the characteristic wavelength). The problem can be essentially simplified by considering self-similar solutions, but also in this case the problem remains quite complex since self-similar solution, generally speaking, is two-dimensional by two spatial coordinates. Therefore this paper considers cases allowing further simplifications and, but as an example, numerical solution of the problem of localized perturbation evolution for the plane case. Density of light fluid which supports heavy one is assumed to equal zero.

Consider potential flow of ideal incompressible fluid for which the equations of mo-

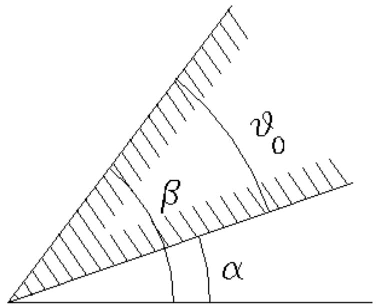

Figure 1

tion are written in the form

$$\Delta\phi = 0 \tag{1}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gy = 0, \tag{2}$$

$$\frac{dX}{dt} = \frac{\partial\phi}{\partial x}\Big|_s, \quad \frac{dY}{dt} = \frac{\partial\phi}{\partial y}\Big|_s, \tag{3}$$

where $\phi(\vec{r},t)$ is the velocity potential ($\vec{v} = \nabla\phi$), p the pressure, ρ the density, g the acceleration of gravity, X, Y the Lagrangian coordinates of the boundary, and the subscript s denotes that the corresponding value is taken on the free surface. Since on the free surface the pressure is p =const, and the potential ϕ is defined within arbitrary function of t, then on the free surface the Eq. (2) takes the form

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gy|_s = 0. \tag{4}$$

The system of Eqs. (1), (3), (4) completely specifies fluid flow.

1 Acute Angles

The simplest is the case of acute angles $\theta_0 < \frac{\pi}{2}$ (Figure 1), considered earlier [6]. Consider this case here for generality. Expand the potential ϕ by powers $x - x_0(t)$, $y - y_0(t)$, where x_0, $y_0(t)$ - the angle vertex coordinates. The velocity potential satisfying

(1) will be written in the form

$$
\begin{aligned}
\phi(x,y,t) = {} & \phi_0(t) + \phi_x(t)(x-x_0) + \phi_y(t)(y-y_0) \\
& + \frac{\phi_{yy}(t)}{2}[(y-y_0)^2 - (x-x_0)^2] \\
& + \phi_{xy}(t)(x-x_0)(y-y_0) + o(r-r_0)^2.
\end{aligned} \tag{5}
$$

By substituting (5) in Eqs. (3), (4) and expanding the surface equation by powers of distance from the corner point we obtain

$$
\begin{aligned}
&\frac{d\phi}{dt} = \frac{\phi_x^2 + \phi_y^2}{2} - gy_0, \\
&\frac{d\phi_x}{dt} = 0, \quad \frac{d\phi_y}{dt} = -g, \\
&\frac{dx_0}{dt} = \phi_x, \quad \frac{dy_0}{dt} = \phi_y,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\frac{d\phi_{yy}}{dt} &= \frac{1-AB}{1+AB}(\phi_{yy}^2 + \phi_{xy}^2), \\
\frac{d\phi_{xy}}{dt} &= \frac{A+B}{1+AB}(\phi_{yy}^2 + \phi_{xy}^2), \\
\frac{dA}{dt} &= 2\phi_{yy}A + \phi_{xy}(1-A^2), \\
\frac{dB}{dt} &= 2\phi_{yy}B + \phi_{xy}(1-B^2),
\end{aligned} \tag{7}
$$

where $A = tg\alpha$, $B = tg\beta$.

The system of Eqs. (6), (7) shows that, first, the flow in the vicinity of acute angles is independent on other fluid mass flow since the characteristics of other region flows are not involved in it. Second, the corner point motion is uniformly accelerated with -g acceleration which results from the pressure gradient equal to zero in it, hence, the corner point freely falls. Third, the corner point motion and angle side rotation are independent since the systems (6) and (7) are disjointed. Fourth, angle rotation and compression depend only on initial conditions and if at the initial time fluid was at rest ($\phi(\vec{r},0) = 0$) then from (7) it follows that the corner will continue neither to rotate nor compress and will freely fall down like a solid whose side angles α and β will be rigidly fixed in space.

2 Self-similarity for angles larger than $\frac{\pi}{2}$

Now consider the case of angles $\theta_0 < \pi/2$. Assume that at the initial time fluid is at rest. Then at the next times while nonlinear terms are yet small and the surface has not had time to displace much the flow in the bulk of the fluid will be determined by the linear problem

$$\Delta\phi = 0, \tag{8}$$
$$\frac{\partial\phi}{\partial t} + gy|_s = 0.$$

Thereby in the potential expansion by powers of the distance from the corner vertex at a sufficient distance from the vertex where non-linear terms are yet unessential, besides the powers (5) the terms of the form

$$Cgt|\vec{r} - \vec{r}_0|^n \cos n\theta, \tag{9}$$

$n = k\pi/\theta_0$, k - integer, θ - angle counted from the angle bisector, will be present and play an essential role. Indeed, such types of terms satisfy Eq. (8), and the lowest power n making the principal contribution to the expansion is $n = \pi/\theta_0$. When $\frac{\pi}{2} < \theta_0 < \pi$ the terms of the form (9) will be larger than quadratic but less than linear in the expansion (5). Therefore, the corner point motion continues to follow the Eq. (6), i.e. constitutes free fall, however in the next approximation the angle motion will be already related to the motion of remaining fluid through the coefficient C, which is determined by solution of the linear problem for entire fluid. When $\theta_0 > \pi$, the terms of the form (9) are larger than linear and the motion induced by them becomes principal. In any case $\theta_0 > \frac{\pi}{2}$ after subtraction of linear terms of the expansion (5) from the potential the motion induced by the asymptotic (9) will be symmetric about the bisector which remains therewith immovable.

In the frame of reference connected to the freely falling point the motion at large distances from the corner point will be determined by the potential (9). The nonlinear terms in the equation (4) and the change of the boundaries (3) will become essential at distances estimated by the relation $r \sim vt \sim \frac{\phi}{r}t \sim Cgt^2 \cdot r^{n-1}$ (from this point on we denote $r = |\vec{r} - \vec{r}_0|$). Thus the non-linear solution determining the angle region flow will be self-similar with the self–similar variable

$$\frac{r^{2-n}}{Cgt^2} \tag{10}$$

and potential asymptotic (9) at large distances.

3 Equations describing flow for angles close to right

Simplifications of the angle region fluid flow problem are possible for angles close to $\frac{\pi}{2}$. In this case one can seek the solution of the problem (in the frame of reference connected to falling angle vertex) in the form

$$\phi = \frac{a(r,t)r^2}{2} \cos(2 + \delta n)\theta, \tag{11}$$

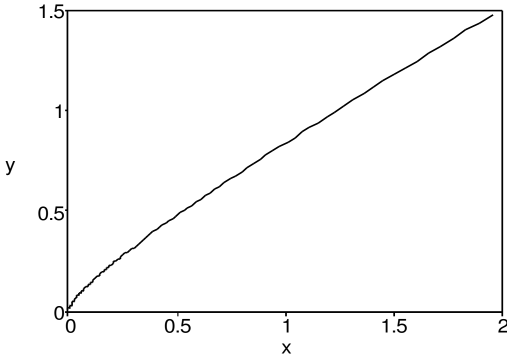

Figure 1:

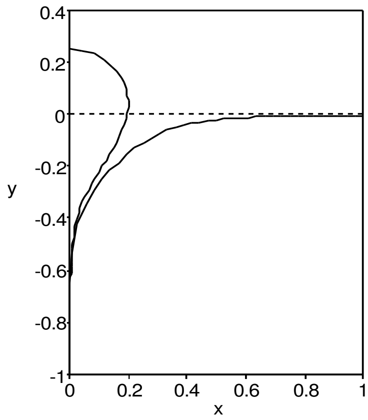

Figure 2:

$$s = rA(r,t),$$

where $a(r,t)$ - the function slowly varying depending on r, $\delta n << 1$, s - surface boundary offset from right angle sides which is taken to be small ($A << 1$, $A > 0$ corresponds to compression). From the Laplace equation (1) it follows that δn must satisfy the relation

$$\delta n = \frac{d\ln a}{d\ln r}. \tag{12}$$

Substituting (11), (12) in Eqs. (3), (4) obtain the equation system

$$A\frac{\partial a}{\partial t} - \frac{\pi}{8}r\frac{\partial^2 a}{\partial r\partial t} + \frac{a^2}{2} = 0, \qquad \frac{\partial A}{\partial t} = a. \tag{13}$$

Thus, if for acute angles the problem reduced to ordinary differential equations, then for the angles close to $\pi/2$ simplification is also possible, though less essentially, so that the two-dimensional problem reduces to the one-dimensional.

4 Self-similar solution for angles slightly larger than right

If an initial angle θ_0 slightly exceeds $\pi/2$, then to describe its collapse one can continue to use Eqs. (13). Let at initial time the angle offset from the right angle be $A = -A_0$, $A_0 << 1$. Then, according to (9), (10) the solution of the problem of collapse of such an angle will be self-similar $a = \frac{2Cgt}{r\frac{8}{\pi}\frac{A_0}{}}u(\xi), A = A_0 w(\xi)$, where $\xi = \frac{Cgt^2}{A_0 r\frac{8A_0}{\pi}}$. Eqs. (13) in these variables transform into the system of ordinary differential equations with initial conditions $u(0) = w(0) = 1$. The angle collapse picture determined by the self-similar solution of these equations is shown in Figure 1 for the case of the angle $\theta_0 = 2\pi/3, A_0 = \pi/12$, (as the problem is symmetric about bisector, a half of the angle is shown, coordinates x, y, are measured in the units $(Cgt^2/2A_0)^{\pi/8A_0})$. If $\xi >> 1$ $w \sim -\xi^{1/3}$, i.e. at large times the angle collapses according to the law $t^{2/3}$ and at large times, owing to weak dependence on r, one can use formulas (7) to describe angle collapse for each r.

5 Localized initial perturbation

The case $\theta_0 = \pi$ (localized perturbation) is special since in this case there are no angle at all and perturbation evolution does not depend on geometry of whole fluid. However, due to especial importance of this case because any perturbation initially occupying a limited region must "forget" its initial form during sufficiently large time period, discuss it in more detail.

We assume the unperturbed fluid surface as horizontal from the equilibrium conditions [8]. From dimension considerations it follows that $\phi = \frac{gt^3}{2}\psi(\frac{2x}{gt^2}, \frac{2y}{gt^2})$. Eqs. (1), (3), (4) for function ψ in the self- similar variables denoted below with the same letters $2x/gt^2$, $y = 2y/gt^2$ are written in the form

$$\Delta\psi = 0, \tag{14}$$

$$3\psi - 2r\nabla\psi + (\nabla\psi)^2 + Y|_s = 0, \tag{15}$$

$$\frac{dY}{dX} = \left.\frac{Y - \frac{\partial\psi}{\partial y}}{X - \frac{\partial\psi}{\partial x}}\right|_s, \tag{16}$$

where $Y(X)$ - the free surface equation (note that in the axisymmetric case self-similar equations describing localized perturbation evolution coincide with Eqs. (14)-(16) for polar coordinates r, z with the difference that Laplacian is written in polar coordinates, z, r are substituted instead of y, x). Since the particles which came to motion at early stages will freely fall at large times it means that the coordinates of the spike tip will be $X = 0$, $Y = -1$. At infinity the fluid is at rest. Then the surface shape must be expected to be like it is shown in Figure 2 (as the problem is symmetric about the replacement $x \to -x$, a half of the surface is shown).

Solution of the problem (14)-(16) is shown in Figure 2. The bubble top coordinate is $Y_0 = 0.248$, the bubble diameter -0.400. Computed bubble diameter agrees with the measured result [5] where it was 0.46. At the same time the computed depth of bubble penetration into fluid is essentially less then experimentally measured: $Y_0 = 0.4$. The same situation takes place for axisymmetric local perturbation evolution: the bubble diameter obtained in numerical computations [6] agrees with the experiment while the depth of its penetration does not. It may be related to measurement errors since in the experiments the height of bubble rising was counted from the surface which being unstable was covered with sheet of spikes and, therefore, its coordinates were hard to estimate.

Thus, for the self-similar solution we have arrived at the surface shape shown in Figure 2. However one can put a question: how will the localized initial perturbation set in the shape of a projection rather than a hole transfer to the shape of Figure 2? One can imagine that the initial projection would yield a spike surrounded from both sides with bubbles and with time thickness of this spike would increase slower (if any) than bubble sizes and in the increasing problem scale the role of the spike will trend to nil with time. Such an evolution dynamics is confirmed by experimental data (see, for example, [5]) which show that adjacent bubbles merge and make up a single bubble.

Opening of angles $\theta_0 > \pi$ (internal) can be studied using the same technique of solving 2D self-similar equations as given above for $\theta_0 = \pi$, but already with account of asymptotic (9). The flow picture may be similar to that shown in Figure 2, i.e. spikes may be generated which will remain in the place of the initial angle vertex.

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