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Scaling Laws of Nonlinear Rayleigh–Taylor and Richtmyer–Meshkov Instabilities*

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Abstract. The nonlinear Rayleigh-Taylor (RT) and Richtmyer-Meshkov (RM) instability are investigated by theoretical models based on single-mode and two-bubble physics at all Atwood numbers (A). A potential-flow treatment of single-mode perturbations shows that the late-time RM single mode velocity decreases as $v \sim \lambda/t$, as well as quantifying a two-bubble overtake interaction ("merger") analogous to that found in the Rayleigh-Taylor instability. Using these results in conjunction with a statistical-mechanics merger model, the late time scaling of the RT bubble and spike fronts, as well as novel scaling laws for the multi-mode RM fronts, are obtained. Multi-mode RT bubble [spike] fronts are found to go as $h_B = 0.05Agt^2$ [$h_S = \alpha_S(A)gt^2$]. The multi-mode RM bubble front is found to go as $h_B = a_B t^{\theta_B}$ where $\theta_B = 0.4$ at all A, while the spike front goes as $h_S = a_S t^{\theta_S}$ where θ_S depends strongly on A. The dependence of a_B and a_S on the initial perturbation was found. These results are in good agreement with full-scale hydrodynamic simulations.

1 Introduction

The Rayleigh-Taylor (RT) instability [1] occurs when a fluid accelerates a heavier fluid, or more generally when a pressure gradient opposes a density gradient. The related Richtmyer-Meshkov (RM) instability [2] occurs when a shock wave passes a perturbed interface between two fluids. These instabilities are of extreme importance in achieving inertial confinement fusion [3]. Under the instabilities, small perturbations on the

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interface between the fluids grow into column-shaped bubbles of light fluid and jet-like spikes of heavy fluids. At late times, a highly disordered mixing zone is formed. Questions of interest include the growth rate of the mixed region (bubble and spike fronts), its large-scale features and the effects of the form of the initial perturbation on the late time behavior.

In the multimode RT instability, both the bubble and spike front penetrations grow as gt^2, where g is the driving acceleration. The large scale structure in the mixed region exhibits a self-similar behavior with this scale [4] [5]. In this case, it is natural that gt^2 is the only dimensional length scale of the problem, after the initial conditions have been forgotten. In contrast, the impulsive nature of the RM instability does not induce such a well defined, self-similar law of fluid interpenetration. Many attempts to experimentally [6], theoretically [7] or numerically [8] [9] [10] derive a simple scaling law of the RM instability did not result in a satisfactory theory that could predict the nonlinear evolution of the mixing zone.

In the following sections we briefly describe our theoretical and numerical study of this system. Detailed information may be found in Ref [12, 13, 14, 15, 16]. We employ a statistical physics approach, modeling the mixing zone caused by these instabilities using effective particles (fluid bubbles and spikes) and their interactions (bubble overtakes or "mergers"). The hydrodynamic behavior of single-mode bubbles and two-bubble competition were calculated using a potential flow model described in Sec 2. The collective behavior of a large bubble ensemble was treated using a bubble-merger model, described in Sec 3. The main characteristic of the bubble front is that it reaches scale-invariant dynamics. This is used in Sec 4 to derive novel scaling laws for the non-linear RT and RM instability growth rates. Additional results on the Rayleigh-Taylor instability in 3D [16], and under a time dependent driving acceleration can be found in elsewhere in the present volume.

2 Potential-flow model of single mode and two-bubble interactions

In order to understand the mixing zone bubble fronts, it is important to understand the single bubble behavior and two-bubble competition interactions. We considered the simple case of an incompressible inviscid fluid accelerated by a much lighter fluid that may be represented by a free boundary with constant supporting pressure. An approximate solution to the resulting potential-flow problem of an array of such bubbles was constructed. The solution generalizes a previous model for single-mode bubbles due to Layzer [18], to include multi-bubble competition in various geometries.

The model, which was presented in Ref [13], is based on the observation that the flow of fluid bubbles is governed by the behavior near their tips. The fluid velocity is given by the gradient of a potential $u = \nabla\phi$. Near the tips, the flow is described by a potential that is a sum of modes that satisfy Laplace's equation in the relevant geometry, with time-dependent coefficients. The interface near the tip of bubble i is

$$z(x,t) = z_{0,i}(t) + z_{1,i}(t)(x - x_i)^2 \tag{1}$$

The interface move with the fluid, as described by the kinematic equation

$$u_z = \partial z/\partial t + u_x \partial z/\partial x \tag{2}$$

evaluated at the interface. The dynamics are given by Bernoulli's equation

$$\partial\phi/\partial t + 1/2(u_x^2 + u_z^2) + gz = const. \tag{3}$$

We expand these equations to second order near the bubble tips.This yields differential equations for the bubble's heights $z_{0,i}(t)$, their curvatures $z_{1,i}(t)$ and the potential amplitudes $a_n(t)$.

The model yields very good results for the well-studied RT case, with the correct linear and early nonlinear evolution, and asymptotic velocity $u = \sqrt{g\lambda/(6\pi)}$ [18]. For the RM case, a new result for the asymptotic single-mode velocity is analytically found: $u = (3\pi)^{-1}\lambda/t$. This result is in agreement with our full-scale simulations [13].

Two-bubble competition was found to be qualitatively similar for both RT and RM instabilities. This represents a coarsening interaction, where large bubbles are formed from the competition between smaller bubbles. The rates at which the merger process occurs were calculated, for a periodic array of bubbles of two different sizes. The basic physics of the process were understood by a simple heuristic model of the balance of buoyancy and kinematic drag forces [15]. The ratio of the drag and buoyancy forces per unit mass goes as the ratio between the bubble area and volume, $S/V \sim 1/\lambda$, and is thus smaller for larger bubbles. Thus, larger bubbles rise faster, leading to the overtake interactions that govern the front evolution.

The potential-flow model was applied also to finite fluid layers and to study aspect-ratio effects in 3D geometries. These results are described in Ref [13].

3 Bubble-merger model for collective behavior of bubble ensemble

The RT mixing zone is topped by column-shaped bubbles of light fluid, rising and competing. At late nonlinear stages, large bubbles rise faster than smaller ones. A bubble adjacent to smaller bubbles expands and accelerates while its neighbors shrink and are swept downstream. This process leads to a constant growth of the surviving bubbles and to an acceleration of the front. This description of the mixing front was pioneered by Sharp and Wheeler (SW), who proposed a model for bubble rise and competition [1, 11]. We presented [12] a bubble-merger model for the collective behavior of bubble ensembles at the fronts, which allows realistic merger rates and treatment of more general flow problems. The model is also simple enough to allow some features to be analytically derived. In the model the bubbles are arranged along a line, and are characterized by their diameters (wavelengths) λ_i. The bubble competition is included by a merger rule: two adjacent bubbles of diameters λ_i and λ_{i+1} merge at a rate $\omega(\lambda_i, \lambda_{i+1})$, forming a new bubble of size $\lambda_i + \lambda_{i+1}$. This represents the surviving bubble expanding to fill the space vacated by the smaller bubble that was washed away from the front. Each bubble rises with a velocity $u(\lambda_i)$ equal to the asymptotic velocity of a periodic array of bubbles with wavelength λ_i. The mean interface hight is found by using the average bubble velocity $dh(t)/dt = \langle u \rangle$.

To analyze the model, we define the size distribution function $g(\lambda, t)d\lambda$ as the number of bubbles in the front with wavelengths within $d\lambda$ of λ. In the mean field approximation, neglecting correlations between neighboring bubbles, we can write an evolution equation for the size distribution:

$$N(t)\partial g(\lambda,t)/\partial t = -2g(\lambda,t)\int_0^\infty g(\lambda',t)\omega(\lambda,\lambda')d\lambda' + \int_0^\lambda g(\lambda-\lambda',t)g(\lambda',t)\omega(\lambda-\lambda',\lambda')d\lambda' \tag{4}$$

where $N(t) = \int_0^\infty g(\lambda, t)d\lambda$ is the total number of bubbles at time t. The first term on the right-hand side of Eq 4 is the rate of elimination of bubbles of wavelength λ by mergers with other bubbles, and the second term is the rate of creation of bubbles of wavelength λ by the merger of two smaller bubbles.

It was instructive to analytically solve the model in a simple case $\omega = const$ [12]. In this case, no correlations are formed and Eq 4 is exact. Its solution was obtained using Laplace transforms. It was shown that the bubble size distribution approaches a scale-invariant form

$$g(\lambda,t) = N(t)\langle\lambda(t)\rangle^{-1} f(\lambda/\langle\lambda(t)\rangle) \tag{5}$$

where $\langle\lambda(t)\rangle$ is the mean bubble diameter at time t. The scaled distribution f is selected out of a family of fixed-points. The selection mechanism is related to the central-limit theorem, since the sizes of bubbles that survive to late times are sums of lengths of many original bubbles [12]. All initial distributions with a finite variance were shown to flow to one fixed-point. “Unphysical” initial distributions with long, power-law tails that have no variance, flow to other scaled distribution functions, each with the same tail as the initial distribution. Thus, for a wide class of initial bubble distributions that includes all physically realizable ones, the dynamics flows to a scale-invariant regime which is independent of initial conditions. This was shown apply to other forms of the merger rate as well, by monte-Carlo simulation of the model.

In order to apply the model to quantitatively derive the front evolution, the single-bubble velocity and two-bubble merger rate ω must be supplied. We obtained these by a potential flow calculation, described in Sec. 2 for $A = 1$, and by numerical simulation of the two-bubble problem for $A < 1$.

4 Scaling laws of nonlinear Rayleigh-Taylor and Richtmyer-Meshkov mixing fronts

In realistic systems, the instabilities develop from noisy initial perturbations, that contain many short-wavelength modes. To find the scaling laws for the multi-mode case, we applied the cellular coarsening model of Sec 3, with the merger rates calculated from the potential flow model. This is described, for the case of a large ratio of fluid densities, in Ref [14]. The scaling of the bubble and spike fronts for all density ratios was derived in [15].

The bubble merger model described above predicts that both the RT and RM front dynamics flow to a scale invariant regime, where the bubble size distribution scales with the average wavelength. The growth rate of the bubble front penetration h_B in this regime was derived. In the RT case, the model results in $h_B = \alpha_B g t^2$. Using the scale-invariant fixed point distribution from the merger model, we find $\alpha_B = 0.05A$. This result is in good agreement with experiments [4] and simulations [11] [4] [5]. The approach to a scale invariant form independent of the detailed initial bubble distribution in the model explains the observed independence of the mixing rate on the initial perturbation [12, 19].

The results for the RM bubble front exhibit new scaling behavior. The bubble front penetration is

$$h_B \sim \lambda_0 (u_0 t/\lambda_0)^{\theta_B} \tag{6}$$

where $\theta_B = 0.4$ at all density ratios,and λ_0 and u_0 are the mean initial wavelength and velocity. This is an important difference from the RT case: in the RM case, the penetration depends on information from the initial perturbation at all times. These predictions are in good agreement with our full scale numerical simulations, using the hydrodynamic code LEEOR-2D [5].

We then considered the spike front, where jets of heavy fluids penetrate the light fluid region. At a very high density ratio between the fluids, $A = 1$, the spike behave as freely falling drops. In the RT case this leads to a spike front penetration that goes as gt^2, the same scaling as the bubble front. But in the RM case, at A=1 the spikes fall at a constant velocity, i.e. as $h_S = a_S t^{\theta_S}$, with $\theta_S = 1$, as compared with $\theta_B = 0.4$ for the bubbles. Thus, in the RM case, *the bubble and spike fronts display different power laws.*

To help understand the spike front behavior, we applied the present method to the spike front, by noting that the dominant spikes visible in the flow have roughly the same periodicity as the dominant bubble structures, and the coarsening of their size can therefore be described by the same merger rate ω. The spike velocity used in the merger model is not its terminal velocity but rather the velocity of a single-mode spike when the amplitude of the corresponding bubble is $h_B/\lambda \approx 0.25$. In the RT case, this yields for the spike front penetration $h_S = \alpha_S(A)gt^2$ with $\alpha_S(A)$ an increasing function of the density ratio, that agrees well with simulation and experimental results. In RM, we find $h_S \sim \lambda_0(u_0t/\lambda_0)^{\theta_S}$, with θ_S going from $\theta_S = 1$ at $A = 1$, to $\theta_S = \theta_B \approx 0.4$ at low A, where the bubble and spike fronts are almost symmetric. These results are in good agreement with our full-scale RM simulations.

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